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Optimum Memoryless Nonlinear Transformation for Weak Narrowband Signals in Noise

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Preface

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The optimum memoryless nonlinear transformation for weak narrowband signals in narrowband noise is derived in terms of the joint probability density function of the noise amplitude and phase modulations. The optimization is in terms of maximizing the magnitude of the deflection of the complex envelope at the nonlinearity output, for small signal inputs of arbitrary characteristics. The optimum nonlinearity is complex, in general, meaning that a phase modulation is superposed, in addition to that present at the input to the nonlinearity. A problem with the behavior of the optimum nonlinearity is traced back to a shortcoming in the approximate analysis, and a method for circumventing the problem is presented. Two methods of treating the spurious weak signal component at the nonlinearity output are considered and compared quantitatively. Finally, the optimum nonlinearity for processing phase differences is derived for a particular model of noise statistics and shown to be closely related to an earlier processor. Keywords:

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14. SUBJECT TERMS (CONT'D.)

Deflection
Complex Envelope
Maximum Deflection
Phase Differences
Spurious Signal
Narrowband

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

t	time
$x(t)$	input waveform, (1),(14)
$s(t)$	input signal waveform, (1)
$n(t)$	input noise waveform, (1)
$g\{x\}$	nonlinearity characteristic, (2),(15)
$y(t)$	nonlinearity output, (2),(15)
$y_1(t)$	signal-plus-noise output, (2),(16)
$y_o(t)$	noise-only output, (2)
overbar	mean value, averaged over noise statistics, (3)
$p(u)$	probability density function of input noise, (3)
var	variance, (8A),(40),(42)
d^2	deflection, (9),(10),(43),(44)
$g_m\{x\}$	optimum nonlinearity, (11),(45)
d_m^2	maximum deflection, (12),(46)

$\tilde{g}_m\{x\}$	modified nonlinearity
$A_x(t)$	input amplitude modulation, (13)
$\phi_x(t)$	input phase modulation, (13)
$h\{A,\phi\}$	nonlinear transformation, (15),(22)
arg	argument of complex number, (15)
$A_s(t)$	signal amplitude modulation, (16)
$\phi_s(t)$	signal phase modulation, (16)
$A_n(t)$	noise amplitude modulation, (16)
$\phi_n(t)$	noise phase modulation, (16)
$p(A,\phi)$	noise joint probability density function, (18)
$p_1(A,\phi)$	partial derivative of p, (25)
$p_2(A,\phi)$	partial derivative of p, (25)
q_1, q_2	auxiliary functions, (31),(47)
z_a, z_b	complex numbers, (33),(35),(36),(57),(58)
Δ	time between samples, (62)
$h\{A,\phi,\bar{A},\bar{æ}\}$	nonlinear transformation, (63)-(65)

OPTIMUM MEMORYLESS NONLINEAR TRANSFORMATION FOR WEAK NARROWBAND SIGNALS IN NOISE

INTRODUCTION

For strong additive noise that is not Gaussian, a nonlinear transformation that suppresses the noise, but passes the signal, is useful in aiding in the detection of weak signals. Here, we will first review the standard memoryless nonlinear transformation of a lowpass real waveform composed of signal-plus-noise or noise-alone and maximize the deflection. A problem arises for the "optimum" nonlinearity, which indicates the possibility of infinite deflection; this behavior is traced to a shortcoming of the approximate analysis, and a method for circumventing it is presented.

Then, we extend these ideas to a narrowband waveform containing both amplitude and phase modulation on the signal as well as the noise. In both cases, a deflection measure, for small input signals with arbitrary characteristics, will be maximized by choice of the arbitrary memoryless nonlinearity characteristic. The presence of a spurious weak signal component at the nonlinearity output will be fully discussed and treated in two different ways. Also, the apparent infinity of the "optimum" nonlinearity and its corresponding deflection will be thoroughly investigated, and a method will be presented for ameliorating the shortcomings of the approximate analysis.

Some of the results of this investigation confirm those in [1,2]. However, we give a full derivation of the method and elaborate at length on how to handle the spurious signal component and anomalous behavior. Additionally, the loss of detectability, caused by the desire to completely suppress the spurious signal component, is evaluated quantitatively.

A complete derivation of the optimum nonlinearity operating in the presence of noise with phase dependence of a particular kind is also presented. The corresponding maximum deflection is derived in terms of the amplitude and phase difference probability density functions. The anomaly for small noise amplitudes is discussed and illustrated by examples.

LOWPASS REAL WAVEFORM

Received real waveform x(t) is composed of signal-plus-noise or noise-alone, where the additive noise n(t) is considerably stronger than the signal and can be non-Gaussian. That is, input

$$x(t) = \left\{ \begin{array}{c} s(t) + n(t) \\ \text{or} \\ n(t) \end{array} \right\} , \tag{1}$$

where s(t) is the signal waveform with arbitrary characteristics. This waveform is passed through arbitrary memoryless nonlinearity g giving output

$$y(t) = g\{x(t)\} = \left\{ \begin{array}{c} g\{s(t)+n(t)\} \\ \text{or} \\ g\{n(t)\} \end{array} \right\} \equiv \left\{ \begin{array}{c} y_1(t) \\ \text{or} \\ y_o(t) \end{array} \right\} . \tag{2}$$

Transformation g need not be analytic.

MEAN OUTPUTS

For a given signal amplitude s(t) at time t, the mean output of the nonlinearity is given by averaging over the noise statistics:*

$$\overline{y_1(t)} = \overline{g\{s(t) + n(t)\}} = \int du\ p(u)\ g\{s(t) + u\} , \tag{3}$$

*Integrals without limits are over the range of nonzero integrand.

where p is the <u>known</u> probability density function of noise n(t) at time t. (We have suppressed any t dependence of p, but this analysis allows for nonstationary additive noise, if need be.) Now let the change of variable, $x = s(t) + u$, be made in (3) to get

$$\overline{y_1(t)} = \int dx\ p(x - s(t))\ g\{x\} = \tag{4}$$

$$\cong \int dx\ [p(x) - s(t)\ p'(x)]\ g\{x\}\ , \tag{5}$$

where we expanded the noise probability density function p about the point x, through linear terms in (weak) signal amplitude s(t).

A note of caution is in order regarding expansion (5). Since x can range over $(-\infty,+\infty)$, we are presuming that probability density function p(x) has a local tangent for <u>all</u> x; that is, p(x) has no discontinuities in slope. If we attempt to employ the following results on a density p(x) that violates this condition, the conclusions may be incorrect and a closer investigation is warranted.

For noise-alone, we set $s(t) = 0$ in (4) to get mean output

$$\overline{y_0(t)} = \int dx\ p(x)\ g\{x\}\ . \tag{6}$$

The difference in mean outputs, that is, signal-present versus signal absent, follows from (5) and (6) as the approximation

$$\overline{y_1(t)} - \overline{y_0(t)} \cong - s(t) \int dx\ p'(x)\ g\{x\}\ . \tag{7}$$

OUTPUT DEFLECTION

At the same time, the variance of the output of the non-linearity for noise-alone is

$$\operatorname{var}(y_o(t)) = \overline{y_o^2(t)} - \overline{y_o(t)}^{\,2} , \tag{8A}$$

where mean-square value

$$\overline{y_o^2(t)} = \overline{g^2\{n(t)\}} = \int dx\ p(x)\ g^2\{x\} . \tag{8B}$$

Combining (6)-(8), we define an output deflection from the nonlinearity g as

$$d^2 = \frac{\left[\overline{y_1(t)} - \overline{y_o(t)}\right]^2}{\operatorname{var}(y_o(t))} =$$

$$= s^2(t)\ \frac{\left[\int dx\ p'(x)\ g\{x\}\right]^2}{\int dx\ p(x)\ g^2\{x\} - \left[\int dx\ p(x)\ g\{x\}\right]^2} . \tag{9}$$

This is an approximation to the deflection since it utilizes (5). Therefore, the following results based on (9) are also approximations.

MAXIMUM DEFLECTION

We would like to maximize this small-signal deflection at the nonlinearity output by choice of the nonlinearity characteristic g. However, two observations should be made. First, the absolute scale of g is immaterial to the deflection; that is, a $g\{x\}$ obviously gives the same deflection d^2 as does $g\{x\}$. Second, an additive constant to g does not affect the deflection; that is, $g\{x\} + b$ gives the same value of d^2 as does $g\{x\}$. This is easily verified by direct substitution of $g\{x\} + b$ for $g\{x\}$ in (9), whereupon b is seen to cancel out everywhere. More generally, nonlinearity $a\, g\{x\} + b$ gives the same deflection as $g\{x\}$.

What this means is that, without loss of generality, we can set the nonlinearity mean output for noise-alone, $\overline{y_o(t)}$, equal to zero, and not detract from the attainable values of deflection criterion (9). So, setting (6) to zero, (9) becomes

$$d^2 = s^2(t)\, \frac{\left[\int dx\; p'(x)\; g\{x\}\right]^2}{\int dx\; p(x)\; g^2\{x\}} \, . \tag{10}$$

But now, by Schwartz's inequality, this ratio is maximized by the optimum memoryless nonlinearity

$$g_m\{x\} = -\frac{p'(x)}{p(x)} = -\frac{d}{dx} \ln p(x) \, . \tag{11}$$

Here, we have taken advantage of the scaling independence in order to supply the factor -1 for convenience. The resulting maximum deflection is

$$d_m^2 = s^2(t) \int dx \, \frac{p'^2(x)}{p(x)} \, . \tag{12}$$

Substitution of optimum nonlinearity (11) into noise-alone mean output (6) immediately yields zero, consistent with the assumption utilized in reducing (9) to (10). We should also notice that signal value s(t) appears as a multiplicative term in general deflections (9) and (10); thus, the optimum nonlinearity g_m can be selected, as in (11), without regard to the particular signal amplitude. It also allows s(t) to be deterministic or random, as the case may be. These approximations are all predicated on the small-signal assumption utilized in (5).

EXAMPLE

As an example of (11) and (12), consider Gaussian noise for which

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left[-\frac{(x-\mu_n)^2}{2\sigma_n^2}\right] \quad \text{for all } x \, .$$

As noted under (5), this probability density function has no discontinuities in slope. Substitution in (11) and (12) gives

$$g_m\{x\} = \frac{x-\mu_n}{\sigma_n^2} \quad \text{for all } x$$

and

$$d_m^2 = \frac{s^2(t)}{\sigma_n^2} \, .$$

Thus the (approximate) optimum nonlinearity is, in fact, linear, and the maximum deflection is the instantaneous signal-to-noise power ratio at the input to the nonlinearity.

By the arguments given in the sequel to (9), we could equally well use nonlinearity

$$\tilde{g}_m\{x\} = x$$

and realize the same maximum deflection d_m^2. This latter nonlinearity, $\tilde{g}_m$, would have a nonzero noise-only mean output, namely μ_n; however, this is not a problem since it is known and could be subtracted from the output of $\tilde{g}_m$, if desired.

In the general case, we can always use (modified) optimum nonlinearity

$$\tilde{g}_m\{x\} = a\ g_m\{x\} + b$$

instead of (11) and still get maximum deflection d_m^2 in (12), where a and b can be chosen for convenience. The major difference is that the noise-only mean output is then b, not zero; however, since b is known, this constitutes no limitation or problem.

PHYSICAL INTERPRETATION

The general situation is depicted in the series of plots in figures 1 - 3. The noise-only output $y_o(t)$ is given on the left side of each plot, while the signal-plus-noise output $y_1(t)$ is given on the right side. The mean of each waveform is indicated by a horizontal dashed line. Figure 1 represents the starting point of the analysis, namely (2). Figure 2 indicates the output waveforms for the case of the optimum nonlinearity g_m in (11), that is, $y_o(t)$ now has zero mean. The nonlinearity g_m maximizes the ratio of the mean of $y_1(t)$ to the standard deviation σ_o of $y_o(t)$. Finally, figure 3 biases both $y_o(t)$ and $y_1(t)$ by constant b and scales both by factor a; it represents the outputs of nonlinearity $\tilde{g}_m$.

Another note of caution is in order relative to approximations (11) and (12). If p(x) is zero at any value of x, the optimum nonlinearity g_m approaches infinity at that point, and the maximum deflection d_m^2 may become infinite as well. This is physically unrealistic and indicates that certain forms of the noise probability density function p are disallowed or that the approximations have gone awry. For example, if p approaches zero at an isolated point, it must do so faster than linearly in order that integral (12) remain finite from the contribution in the neighborhood of that point. However, integral (12) is itself an approximation and must also be investigated more closely.

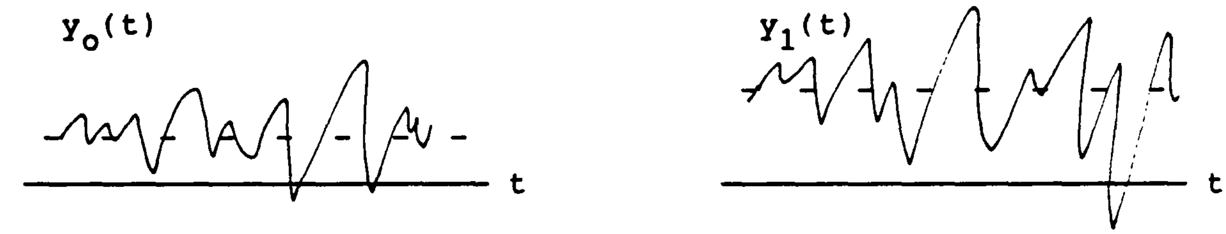

Figure 1. Outputs from General Nonlinearity g

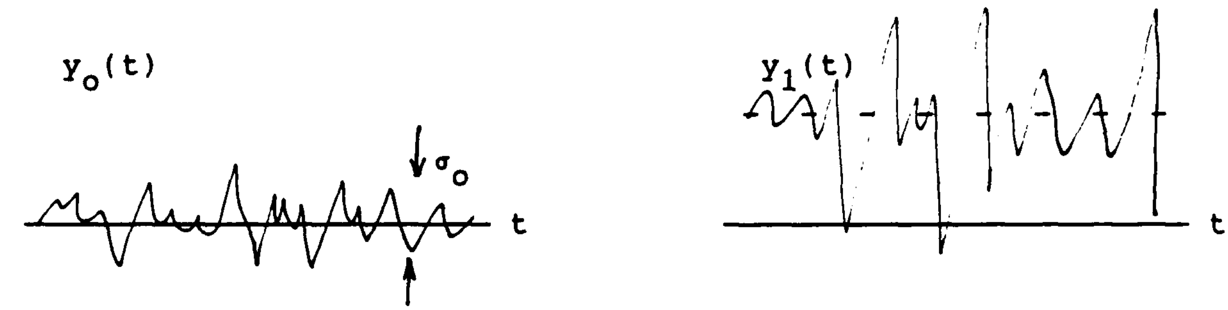

Figure 2. Outputs from Optimum Nonlinearity g_m

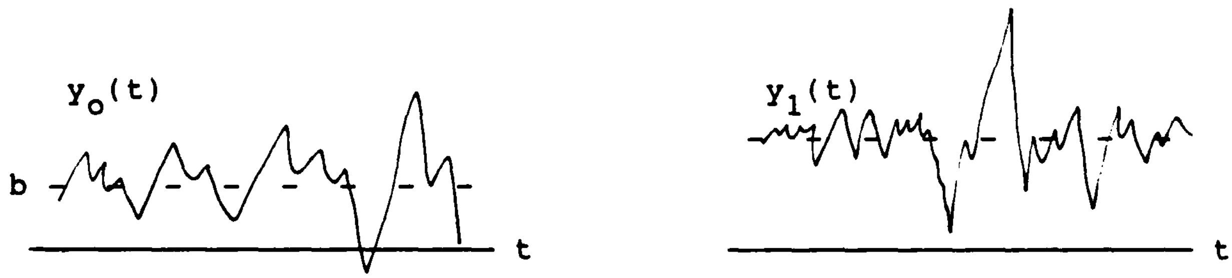

Figure 3. Outputs from Modified Optimum Nonlinearity $\tilde{g}_m$

This possibility of infinite deflection is not entirely due to inadequacy of the approximation utilized in (5) et seq. In general, if we use (4), (6), (8), and the upper line of (9), we have, for any g, exact deflection

$$\bar{d}^2(s) = \frac{\left[\int dx\ [p(x-s) - p(x)]\ g\{x\}\right]^2}{\int dx\ p(x)\ g^2\{x\}} ,$$

where we have replaced s(t) by s for notational brevity. Now suppose that we consider signal value s known and that nonlinearity g can be chosen with this knowledge. The optimum nonlinearity is then (with no approximations)

$$\bar{g}_e\{x;s\} = \frac{p(x-s) - p(x)}{p(x)} = \frac{p(x-s)}{p(x)} - 1.$$

Although it is physically unrealistic to presume signal value s known, this approach is informative in that it pinpoints the source and rate of approach of the infinities. It is immediately seen that if p(x) approaches zero somewhere, then $\bar{g}_e\{x;s\}$ approaches infinity at that x value (unless s = 0). The corresponding maximum deflection for nonlinearity $\bar{g}_e\{x;s\}$ is

$$\bar{d}_e^2(s) = \int dx\ \frac{\left[p(x-s) - p(x)\right]^2}{p(x)} = \int dx\ \frac{p^2(x-s)}{p(x)} - 1 ,$$

and will remain finite only if p approaches zero <u>less</u> fast than linearly. Thus, the condition on the rate of approach of p to zero, in order to maintain finite deflection, is <u>reversed</u> from the conclusion above based on approximation (5). This reversal

is very important to know about, but the possibility of infinite deflection still remains.

If we now make the small-signal assumptions on the exact results above, we find

$$\bar{g}_e\{x;s\} \sim -s\,\frac{p'(x)}{p(x)} = s\,g_m\{x\}\ ,$$

$$\bar{d}_e^2(s) \sim s^2 \int dx\,\frac{p'^2(x)}{p(x)} = d_m^2\ ,$$

consistent with (11) and (12), respectively. (The multiplicative factor of s in $\bar{g}_e\{x;s\}$ is merely an irrelevant scale factor, as far as the deflection is concerned.) No nonlinearity can outperform $\bar{d}_e^2(s)$ for any signal value s, since the latter result allows for use of knowledge of p(x) as well as value s. So if d_m^2 in (12) gives a result larger than $\bar{d}_e^2(s)$, it means that the approximation giving rise to (12) was faulty. In that case, we should revert to exact deflection $\bar{d}^2(s)$ and substitute the particular nonlinearity g being employed. For example, if approximate optimum nonlinearity g_m in (11) is utilized, the corresponding deflection is found to be

$$\bar{d}_m^2(s) = \frac{\left[\int dx\; p(x-s)\; p'(x)/p(x)\right]^2}{\int dx\; p'^2(x)/p(x)}\ .$$

Of course, $\bar{d}_m^2(s) \le \bar{d}_e^2(s)$ in all cases.

Some examples are useful at this point. For Gaussian noise, we have

$$p(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) \quad \text{for all } x \ ,$$

$$g_m\{x\} = -\frac{p'(x)}{p(x)} = x \ ,$$

$$d_m^2 = s^2 \int dx \, \frac{p'^2(x)}{p(x)} = s^2 \ ,$$

$$\bar{d}_m^2(s) = s^2 \ ,$$

$$\bar{g}_e\{x;s\} = \exp\left(sx - \tfrac{1}{2}s^2\right) - 1 \sim sx = s \, g_m\{x\} \ ,$$

$$\bar{d}_e^2(s) = \exp(s^2) - 1 \ \sim s^2 = d_m^2 \ .$$

The quantity $\bar{d}_e^2(s)$ is larger than d_m^2 for all s (except s = 0). All of these results are self consistent.

For exponential noise,

$$p(x) = \frac{1}{2} \exp(-|x|) \quad \text{for all } x \ ,$$

$$g_m\{x\} = -\frac{p'(x)}{p(x)} = \operatorname{sgn}(x) \ ,$$

$$d_m^2 = s^2 \int dx \, \frac{p'^2(x)}{p(x)} = s^2 \ ,$$

$$\bar{d}_m^2(s) = \left[1 - \exp(-|s|)\right]^2 \sim s^2 - |s|^3 \ ,$$

$$\bar{g}_e\{x;s\} = \exp(-|x-s| + |x|) - 1 \ ,$$

$$\bar{d}_e^2(s) = \frac{2}{3} \exp(|s|) + \frac{1}{3} \exp(-2|s|) - 1 \sim s^2 - \frac{1}{3} |s|^3 \ .$$

Now, $\bar{d}_e^2(s)$ is less than d_m^2 for $0 < |s| < 2.07$; thus the approximation d_m^2 is somewhat optimistic and $\bar{d}_m^2(s)$ should be used instead. A sketch of nonlinearity $\bar{g}_e\{x;s\}$ reveals that it resembles

s sgn(x), especially for small s. This example is consistent except for d_m^2, despite the fact that $p'(x)$ is discontinuous at the one point x = 0.

For a noise probability density function with a zero (at the origin for convenience), we have, for example,

$$p(x) = a\ |x|^{\nu} \exp(-|x|^{\mu})\ , \qquad \nu > 0, \quad \mu > 0,$$

$$g_m\{x\} = -\frac{p'(x)}{p(x)} \sim -\frac{\nu}{x} \quad \text{as } x \to 0\ ,$$

$$d_m^2 = s^2 \int dx\ \frac{p'^2(x)}{p(x)} < \infty \quad \text{if } \nu > 1\ ,$$

$$\bar{d}_m^2(s) = 0 \quad \text{if } \nu < 1\ ,$$

$$\bar{g}_e\{x;s\} = \frac{p(x-s)}{p(x)} - 1 \sim \frac{p(-s)}{a\ |x|^{\nu}} \quad \text{as } x \to 0\ ,$$

$$\bar{d}_e^2(s) < \infty \quad \text{if } \nu < 1\ .$$

As anticipated, the condition for finite deflection is reversed in the exact result ($\nu < 1$) versus the approximation ($\nu > 1$). In addition, the rate of growth of the optimum nonlinearity near x = 0 is milder for the exact result and of a very different character. The reason that $\bar{d}_m^2(s)$ is zero for $\nu < 1$ is that $p'(x)/p(x) \sim 1/x$, whereas $p'^2(x)/p(x) \sim x^{\nu-2}$ as $x \to 0+$; thus, the integrand of the denominator of $\bar{d}_m^2(s)$ has a higher-order singularity at x = 0 and approaches infinity at a faster rate than the numerator. This example exemplifies the need for close scrutiny of a noise probability density function which has a zero value anywhere in its range.

NARROWBAND WAVEFORM

The available input waveform of interest in this section has the form

$$A_x(t) \cos[2\pi f_o t + \phi_x(t)] \ , \tag{13}$$

where f_o is the known center frequency, and $A_x(t)$ and $\phi_x(t)$ are the lowpass amplitude and phase modulations, respectively. The complex envelope of this waveform, which can be easily extracted from (13), is

$$x(t) = A_x(t) \exp[i\phi_x(t)] \ . \tag{14}$$

We will allow an arbitrary complex memoryless nonlinear transformation h of $A_x(t)$ and $\phi_x(t)$; that is, the output is modified complex envelope

$$y(t) = h\{A_x(t),\phi_x(t)\} = h\{|x(t)|, \arg(x(t))\} \equiv g\{x(t)\} \ , \tag{15}$$

where g is an arbitrary complex function of complex argument x(t). Transformation g need not be analytic.

The input x(t) to nonlinearity g is composed of signal-plus-noise (or noise-alone); thus, we can express the signal-plus-noise output as

$$y_1(t) = g\{A_s(t) \exp[i\phi_s(t)] + A_n(t) \exp[i\phi_n(t)]\} \ , \tag{16}$$

where $A_s(t)$ and $\phi_s(t)$ are the arbitrary input signal amplitude and phase modulations, while $A_n(t)$ and $\phi_n(t)$ are the input noise amplitude and phase modulations, respectively. We presume a low

input signal-to-noise ratio; that is,

$$A_s^2(t) \ll \overline{A_n^2(t)} \quad . \tag{17}$$

However, there are no limitations on the sizes of phase modulations $\phi_s(t)$ and $\phi_n(t)$, nor on the signal characteristics. The joint probability density function of the noise amplitude and phase modulations is presumed known; that is, $p(A_n,\phi_n)$ is given. (Again, although we suppress any t dependence of p, the following analysis allows for nonstationary noise simply by reinstating any t dependence in p.)

MEAN OUTPUTS

For given signal amplitude and phase modulations $A_s(t)$ and $\phi_s(t)$, the complex mean output from the nonlinearity g is, from (16),

$$\overline{y_1(t)} = \iint dA_n \; d\phi_n \; g\{A_s(t)\exp[i\phi_s(t)]+A_n\exp[i\phi_n]\} \; p(A_n,\phi_n) . \tag{18}$$

We now make the change of variables

$$A \exp[i\phi] = A_s(t) \exp[i\phi_s(t)] + A_n \exp[i\phi_n] \tag{19}$$

in (18). The Jacobian of this two-dimensional transformation is derived in appendix A; it is

$$\frac{\partial(A_n,\phi_n)}{\partial(A,\phi)} = \frac{A}{A_n} = \frac{A}{|A \exp[i\phi] - A_s(t) \exp[i\phi_s(t)]|} \quad . \tag{20}$$

At this point, for notational convenience, we will suppress the t dependence of A_s and ϕ_s; this time dependence will be reestablished after all the following mathematical manipulations have been completed. The use of (19) and (20) converts (18) into the following exact result for the complex mean output

$$\overline{y_1(t)} = \iint dA\, d\phi\, \frac{A}{|A \exp(i\phi) - A_s \exp(i\phi_s)|}\, g\{A \exp(i\phi)\} \times$$

$$\times\, p\Big(|A \exp(i\phi) - A_s \exp(i\phi_s)|,\ \arg\{A \exp(i\phi) - A_s \exp(i\phi_s)\}\Big). \quad (21)$$

This expression could be written in an entirely equivalent form by replacing $g\{A \exp(i\phi)\}$ with $h\{A,\phi\}$; that is, from (14) and (15),

$$h\{A,\phi\} = g\{A \exp(i\phi)\} \quad \text{for all } A,\phi \ . \quad (22)$$

This latter form, in terms of h, more clearly accents that a completely arbitrary transformation of A and ϕ is allowed; however, since g is arbitrary, the same is true of the form $g\{A \exp(i\phi)\}$, which is used henceforth.

SMALL INPUT SIGNAL-TO-NOISE RATIO

We now make use of the low input signal-to-noise ratio assumption, (17), by expanding (21) through first-order terms in the signal amplitude modulation A_s. Since

$$|1 - \varepsilon - i\delta| = \sqrt{(1 - \varepsilon)^2 + \delta^2} \sim 1 - \varepsilon ,$$

and

$$\arg(1 - \varepsilon - i\delta) \sim - \delta ,$$

through linear terms in real variables ε and δ, then

$$\begin{aligned} &|A \exp(i\phi) - A_s \exp(i\phi_s)| \sim A - A_s \cos(\phi_s - \phi) \quad \text{for } A_s << A , \\ &\arg\{A \exp(i\phi) - A_s \exp(i\phi_s)\} \sim \phi - \frac{A_s}{A} \sin(\phi_s - \phi) \quad \text{for } A_s << A . \end{aligned} \tag{23}$$

Substitution in (21) yields

$$\overline{y_1(t)} \sim \iint dA\, d\phi \frac{A}{A - A_s \cos(\phi_s - \phi)} g\{A \exp(i\phi)\} \times$$

$$\times\, p\left(A - A_s \cos(\phi_s - \phi),\ \phi - \frac{A_s}{A} \sin(\phi_s - \phi)\right) . \tag{24}$$

Define

$$p_1(A,\phi) = \frac{\partial}{\partial A} p(A,\phi) , \quad p_2(A,\phi) = \frac{\partial}{\partial \phi} p(A,\phi) , \tag{25}$$

where it is presumed that these derivatives of p exist. Then the term in (24) involving joint probability density function p can be expanded as

$$p(A,\phi) - A_s \cos(\phi_s - \phi)\, p_1(A,\phi) - \frac{A_s}{A} \sin(\phi_s - \phi)\, p_2(A,\phi)\ , \tag{26}$$

to linear terms in A_s. Coupled with

$$\frac{A}{A - A_s \cos(\phi_s - \phi)} \sim 1 + \frac{A_s}{A} \cos(\phi_s - \phi) \quad \text{for } A_s << A\ , \tag{27}$$

(24) develops into

$$\overline{y_1(t)} \sim \iint dA\, d\phi \left[1 + \frac{A_s}{A} \cos(\phi_s - \phi)\right] g\{A \exp(i\phi)\} \times$$

$$\times \left[p(A,\phi) - A_s \cos(\phi_s - \phi)\, p_1(A,\phi) - \frac{A_s}{A} \sin(\phi_s - \phi)\, p_2(A,\phi)\right] =$$

$$\sim \iint dA\, d\phi\, g\{A \exp(i\phi)\} \left[p(A,\phi) + \frac{A_s}{A} \cos(\phi_s - \phi)\, p(A,\phi) - \right.$$

$$\left. - A_s \cos(\phi_s - \phi)\, p_1(A,\phi) - \frac{A_s}{A} \sin(\phi_s - \phi)\, p_2(A,\phi)\right]\ , \tag{28}$$

through linear terms in A_s. There is no presumption about the form of nonlinearity g in these expressions.

There is a fundamental flaw in the use of approximations (23) and (27) in integral (21). The approximations specifically require that $A > A_s$, yet they are used in end result (28) all the way down to $A = 0$. The results in (23) should be augmented with

$$\left.\begin{aligned} |A \exp(i\phi) - A_s \exp(i\phi_s)| &\sim A_s \\ \arg\{A \exp(i\phi) - A_s \exp(i\phi_s)\} &\sim \phi_s + \pi \end{aligned}\right\} \quad \text{for } A << A_s\ .$$

This would not only eliminate the troublesome 1/A dependencies for small A in (23)-(28), but in fact convert the Jacobian to a linear A dependence for small A, a very marked change.

The reason we do not incorporate this behavior is that it would greatly complicate (28), and the intermediate range, $A \approx A_s$, would still not be covered. What this means is that we can anticipate some problems with approximation (28) and further results based on (28), for small A; in fact, we must be willing to modify or discard the 1/A dependency in some cases and ranges, since it is based upon an invalid approximation. We will return to this point later and elaborate in more detail.

The mean output from complex nonlinearity g, for noise-only, is available directly from exact result (21) by setting $A_s = 0$:

$$\overline{y_o(t)} = \iint dA\, d\phi\, g\{A \exp(i\phi)\}\, p(A,\phi) \ . \tag{29}$$

DIFFERENCE IN MEAN OUTPUTS

The difference in complex mean outputs from nonlinearity g, for signal-present versus signal-absent, is then available from (28) and (29) as

$$\overline{y_1(t)} - \overline{y_o(t)} = - A_s \iint dA\, d\phi\, g\{A \exp(i\phi)\} \times$$

$$\times \left[\cos(\phi_s - \phi)\, q_1(A,\phi) + \sin(\phi_s - \phi)\, q_2(A,\phi)\right] , \qquad (30)$$

where, using (25), the real quantities

$$q_1(A,\phi) = A \frac{\partial}{\partial A} \left(\frac{p(A,\phi)}{A}\right) , \quad q_2(A,\phi) = \frac{\partial}{\partial \phi} \left(\frac{p(A,\phi)}{A}\right) . \qquad (31)$$

The approximate result in (30) is similar to that in (7) for the lowpass real case in that signal amplitude A_s appears multiplicatively as a linear factor. However, (30) is still complicated by the appearance of signal phase ϕ_s inside the integrals. If we expand the cos and sin terms in (30), we find

$$\overline{y_1(t)} - \overline{y_o(t)} = - A_s (\cos\phi_s\, z_a + \sin\phi_s\, z_b) , \qquad (32)$$

where complex numbers (due to the allowed complexity of g)

$$z_a = \iint dA\, d\phi\, g\{A \exp(i\phi)\} \left[\cos\phi\, q_1(A,\phi) - \sin\phi\, q_2(A,\phi)\right] ,$$

$$z_b = \iint dA\, d\phi\, g\{A \exp(i\phi)\} \left[\sin\phi\, q_1(A,\phi) + \cos\phi\, q_2(A,\phi)\right] . \qquad (33)$$

An alternative form of (32) is more useful; specifically, the difference of mean outputs can be expressed as

$$\overline{y_1(t)} - \overline{y_o(t)} = -\frac{1}{2} A_s \exp(i\phi_s)\,(z_a - iz_b) - \frac{1}{2} A_s \exp(-i\phi_s)\,(z_a + iz_b) = \tag{34}$$

$$= -\frac{1}{2} A_s(t)\,\exp[i\phi_s(t)]\,(z_a - iz_b) - \frac{1}{2} A_s(t)\,\exp[-i\phi_s(t)]\,(z_a + iz_b),$$

where we have reestablished the time dependence of the signal amplitude and phase modulations $A_s(t)$ and $\phi_s(t)$, respectively. The two complex numbers in (34) can be written as

$$z_a - iz_b = \iint dA\, d\phi\; g\{A \exp(i\phi)\}\, \exp(-i\phi)\, [q_1(A,\phi) - iq_2(A,\phi)], \tag{35}$$

$$z_a + iz_b = \iint dA\, d\phi\; g\{A \exp(i\phi)\}\, \exp(+i\phi)\, [q_1(A,\phi) + iq_2(A,\phi)]. \tag{36}$$

The leading term in (34) contains a replica of the input signal to the nonlinearity g, namely,

$$A_s(t)\, \exp[i\phi_s(t)]\ , \tag{37}$$

and will be called the desired signal component in the difference of mean outputs. The remaining signal-dependent term in (34), namely,

$$A_s(t)\, \exp[-i\phi_s(t)]\ , \tag{38}$$

is of no interest since it has a distorted phase modulation. That is, (38) will not correlate with the local reference, since the latter has exactly the same form (38). Henceforth, we simply

ignore the extra term (38) in difference (34). Thus, for the desired signal component, we have

$$\left(\overline{y_1(t)} - \overline{y_o(t)}\right)_{\text{desired}} = -\frac{1}{2} A_s(t) \exp[i\phi_s(t)] \, (z_a - iz_b) \, , \quad (39)$$

where the complex number $z_a - i\, z_b$ is given by (35).

The simplest example of this behavior is furnished by the nonanalytic nonlinearity

$$g\{z\} = |z|^2 \, ,$$

for which the output is

$$y_1(t) = |A_s(t) \exp[i\phi_s(t)] + A_n(t) \exp[i\phi_n(t)]|^2 =$$

$$= A_n^2(t) + A_n(t) \exp[-i\phi_n(t)] \, A_s(t) \exp[i\phi_s(t)] \, +$$

$$+ A_n(t) \exp[i\phi_n(t)] \, A_s(t) \exp[-i\phi_s(t)] + A_s^2(t) \, .$$

Both types of signal terms, (37) and (38), are exhibited here.

VARIANCE OF OUTPUT

At the same time, the variance of the output of the nonlinearity, for noise-alone, is

$$\mathrm{var}(y_o(t)) = \overline{\left| y_o(t) - \overline{y_o(t)} \right|^2} =$$

$$= \overline{\left| y_o(t) \right|^2} - \left| \overline{y_o(t)} \right|^2 , \qquad (40)$$

where $y_o(t)$ is available from (16) by setting $A_s(t) = 0$:

$$y_o(t) = g\{A_n(t) \exp[i\phi_n(t)]\} . \qquad (41)$$

There follows immediately the exact result

$$\mathrm{var}(y_o(t)) = \overline{\left| g\{A_n(t) \exp[i\phi_n(t)]\} \right|^2} - \left| \overline{g\{A_n(t) \exp[i\phi_n(t)]\}} \right|^2 =$$

$$= \iint dA \, d\phi \, |g\{A \exp(i\phi)\}|^2 \, p(A,\phi) -$$

$$- \left| \iint dA \, d\phi \, g\{A \exp(i\phi)\} \, p(A,\phi) \right|^2 , \qquad (42)$$

where $p(A,\phi)$ is again the joint probability density function of the noise amplitude and phase modulations.

OUTPUT DEFLECTION

We are now prepared to define an output deflection from nonlinearity g (analogous to (9)) as

$$d^2 = \frac{\left|\left(\overline{y_1(t)} - \overline{y_o(t)}\right)_{\text{desired}}\right|^2}{\operatorname{var}(y_o(t))} =$$

$$= \frac{1}{4} A_s^2(t) \frac{\left|\iint dA\, d\phi\, g\{A \exp(i\phi)\} \exp(-i\phi)\, [q_1(A,\phi) - iq_2(A,\phi)]\right|^2}{\iint dA\, d\phi\, \left|g\{\ \}\right|^2 p(A,\phi) - \left|\iint dA\, d\phi\, g\{\ \}\, p(A,\phi)\right|^2},$$

(43)

where we used (39), (35), and (42). Since (30)-(39) are based on approximation (28), deflection (43) is likewise approximate.

MAXIMUM DEFLECTION

We would like to maximize this small-signal deflection (43) at the nonlinearity output, by choice of the nonlinearity characteristic g. However, as in the sequel to (9), two important observations must be made. First, the absolute scale of g is obviously immaterial to the value of d^2. Second, an additive complex constant to g does not affect d^2; this last property is derived in appendix B. Thus, nonlinearity $a\, g\{A \exp(i\phi)\} + b$ gives the same deflection as $g\{A \exp(i\phi)\}$, where a and b are arbitrary complex constants.

What this means is that, without loss of generality, we can set the complex nonlinearity mean output for noise-alone, $\overline{y_o(t)}$,

equal to zero, and not detract from the attainable values of deflection criterion (43). So, setting (29) to zero, (43) becomes

$$d^2 = \frac{A_s^2(t)}{4} \frac{\left|\iint dA\, d\phi\; g\{A \exp(i\phi)\}\, \exp(-i\phi)\, [q_1(A,\phi) - iq_2(A,\phi)]\right|^2}{\iint dA\, d\phi\; \left|g\{A \exp(i\phi)\}\right|^2 p(A,\phi)} . \tag{44}$$

But now, since g is completely arbitrary, by Schwartz's inequality, this ratio is maximized by the (approximate) optimum memoryless nonlinearity

$$g_m\{A \exp(i\phi)\} = h_m\{A,\phi\} = -\frac{\exp(i\phi)\,\left(q_1(A,\phi) + iq_2(A,\phi)\right)}{p(A,\phi)} =$$

$$= -\frac{\exp(i\phi)}{p(A,\phi)} \left[A \frac{\partial}{\partial A}\left(\frac{p(A,\phi)}{A}\right) + i \frac{\partial}{\partial \phi}\left(\frac{p(A,\phi)}{A}\right)\right] =$$

$$= -\exp(i\phi)\left[\frac{\partial}{\partial A} \ln\left(\frac{p(A,\phi)}{A}\right) + \frac{i}{A} \frac{\partial}{\partial \phi} \ln\{p(A,\phi)\}\right] , \tag{45}$$

where we used (22) and (31). Here, we also have taken advantage of the scaling independence in order to supply the factor -1 for convenience. This result agrees with [1; (9)].

The resulting maximum deflection is

$$d_m^2 = \frac{1}{4} A_s^2(t) \iint dA\, d\phi\, \frac{\left|q_1(A,\phi) + iq_2(A,\phi)\right|^2}{p(A,\phi)} =$$

$$= \frac{1}{4} A_s^2(t) \iint dA\, d\phi\, \frac{q_1^2(A,\phi) + q_2^2(A,\phi)}{p(A,\phi)} , \tag{46}$$

where q_1 and q_2 are available from (31) as

$$q_1(A,\phi) = A \frac{\partial}{\partial A}\left(\frac{p(A,\phi)}{A}\right) \ , \quad q_2(A,\phi) = \frac{\partial}{\partial \phi}\left(\frac{p(A,\phi)}{A}\right) \ . \qquad (47)$$

Whether the phase term, $q_2(A,\phi)/p(A,\phi)$, in optimum nonlinearity (45) is important or not can be ascertained from (46) by evaluating it with and without the q_2^2/p term present. Some limitations of approximations (45) and (46) concerning the 1/A dependencies are given in appendix C, as well as an alternative approach.

Substitution of optimum nonlinearity (45) into noise-alone mean output (29) immediately yields the conjugate of the integral in (B-1), which is shown to be identically zero in (B-4). This is consistent with the nonrestrictive assumption utilized in reducing (43) to (44). It should also be noted that signal amplitude $A_s(t)$ appears as a multiplicative term in general deflections (43) and (44); thus, the optimum nonlinearity g_m can be selected, as in (45), without regard to the particular signal amplitude. It also allows $A_s(t) \exp[i\phi_s(t)]$ to be deterministic or random, as the case may be. This is all predicated on the small signal assumption utilized in (22)-(28). The actual output time waveform from the optimum nonlinearity g_m in (45) is obtained by replacing argument $A \exp(i\phi)$ with $A_s(t) \exp[i\phi_s(t)] + A_n(t) \exp[i\phi_n(t)]$; see (16).

PHYSICAL INTERPRETATION

A physical interpretation of what is taking place in this complex envelope case is given in figures 4 - 6. The noise-only output $y_o(t)$ is plotted on the left side of each figure as a complex point in the plane, which moves as time progresses. The noise-only mean output for arbitrary nonlinearity g is indicated by a dashed arrow in figure 4 to the complex point $\overline{y_o(t)}$ given by (29). For any nonlinearity g selected, this is a <u>known</u> point since joint probability density function $p(A,\phi)$ is known.

When signal is also present, the situation for output $y_1(t)$ is depicted on the right side of each figure. The mean output for arbitrary nonlinearity g is indicated by a dashed arrow in figure 4 to the complex point $\overline{y_1(t)}$ given by (21) or (28). This, too, is a known point for specified g, A_s, ϕ_s.

When we choose the class of nonlinearities g that have zero-mean noise-only output $\overline{y_o(t)}$, as done in (43)-(44), we are taking advantage of knowledge of these locations and the situation is as shown in figure 5. The mean location of complex waveform $y_o(t)$ is now at the origin of coordinates and its standard deviation from the origin is indicated by σ_o. Then the plot of $y_1(t)$ appears on the right side of figure 5, where the dashed arrow is drawn to the point $\overline{y_1(t)}$. When we maximize the deflection criterion d^2 in (44), we are maximizing the ratio of the length of the arrow on the right side to the standard deviation σ_o on the left side. Physically, we are trying to make

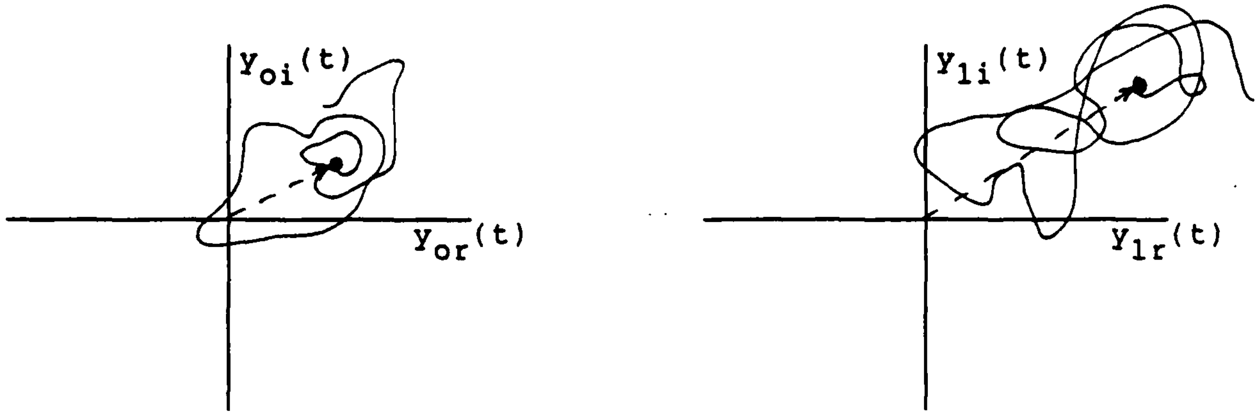

Figure 4. Outputs from General Nonlinearity g

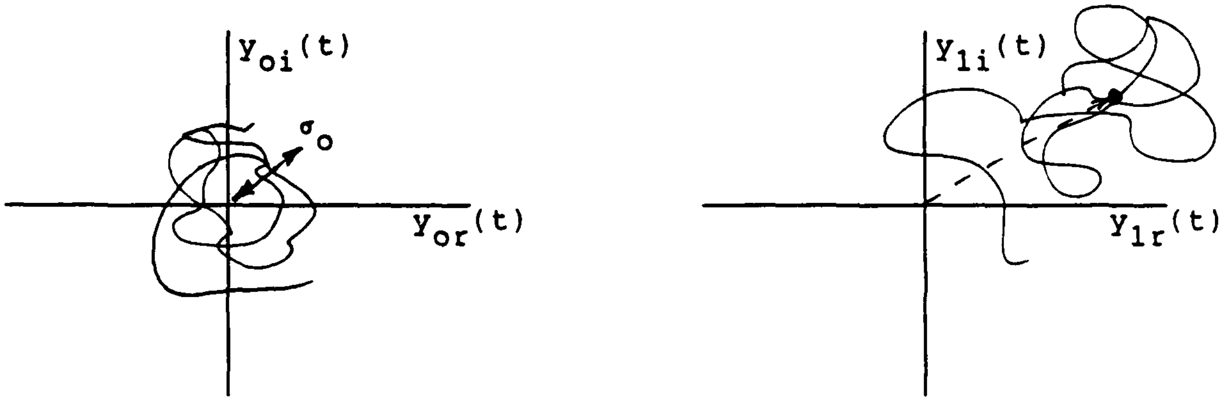

Figure 5. Outputs from Optimum Nonlinearity g_m

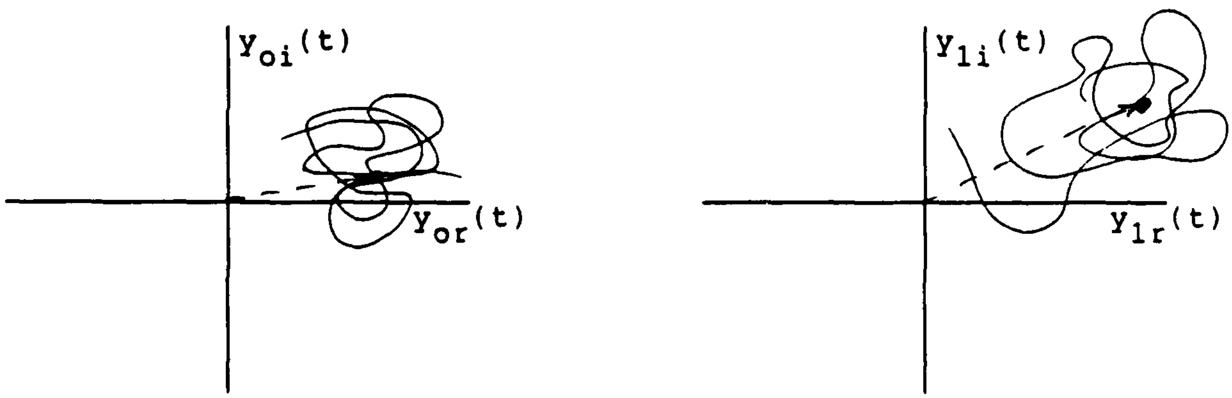

Figure 6. Outputs from Modified Optimum Nonlinearity $\tilde{g}_m$

the signal-present average distance from the origin as large as possible relative to the signal-absent deviations from the origin.

Finally, figure 6 represents the outputs when the optimum nonlinearity is scaled and biased by an arbitrary factor and additive constant. Both $y_o(t)$ and $y_1(t)$ are similarly scaled and shifted, but the maximum deflection is unchanged.

EXAMPLES

As an example of (45) and (46), suppose the narrowband noise is zero-mean Gaussian; then

$$p(A,\phi) = \frac{A}{2\pi\sigma_n^2} \exp\left(-\frac{A^2}{2\sigma_n^2}\right) \quad \text{for } A > 0,\ |\phi| < \pi , \tag{48}$$

giving

$$g_m\{A \exp(i\phi)\} = \frac{1}{\sigma_n^2} A \exp(i\phi) \tag{49}$$

and

$$d_m^2 = \frac{A_s^2(t)/2}{\sigma_n^2} . \tag{50}$$

Thus, the optimum nonlinearity in (49) is linear and the maximum deflection is the signal-to-noise power ratio at the input to the nonlinearity. A more thorough analysis of this example is given in appendix C.

By the arguments given in the sequel to (43), we could equally well use nonlinearity

$$\tilde{g}_m\{A\ \exp(i\phi)\} = A\ \exp(i\phi)$$

and realize the same maximum deflection d_m^2 in (50). In the <u>general</u> case, we can always use (modified) optimum nonlinearity

$$\tilde{g}_m\{A\ \exp(i\phi)\} = a\ g_m\{A\ \exp(i\phi)\} + b$$

instead of (45), and still realize maximum deflection d_m^2 in (46), where complex constants a and b can be chosen for convenience. The major difference is that the noise-only mean output is then b, not zero; however, since b is known, this constitutes no limitation or problem. This case is depicted in figure 6.

As a second (more general) example, if the noise amplitude and phase modulations are statistically independent, then

$$p(A,\phi) = p_a(A)\ p_b(\phi)\ , \tag{51}$$

and (45) reduces to

$$g_m\{A\ \exp(i\phi)\} = h_m\{A,\phi\} = -\exp(i\phi)\left[\frac{d}{dA}\ \ln\left(\frac{p_a(A)}{A}\right) + i\ \frac{1}{A}\ \frac{p_b'(\phi)}{p_b(\phi)}\right]\ . \tag{52}$$

This result agrees with [1; (10) and (16)]. The maximum deflection, (46), is

$$d_m^2 = \frac{1}{4} A_s^2(t) \left\{ \int dA \frac{\left[p_a'(A) - p_a(A)/A\right]^2}{p_a(A)} + \right.$$

$$\left. + \int dA \frac{p_a(A)}{A^2} \int d\phi \frac{p_b'^2(\phi)}{p_b(\phi)} \right\} . \quad (53)$$

The relative importance of the $p_b'(\phi)$ term can be easily ascertained from here. The 1/A dependencies are thoroughly discussed in appendix C.

DISCUSSION OF IMAGINARY TERM

In general, for the optimum nonlinearity in (45), the $\exp(i\phi)$ factor indicates a replication of the phase-modulation at its input; that is, from (16),

$$\arg\{A_s(t) \exp[i\phi_s(t)] + A_n(t) \exp[i\phi_n(t)]\} \quad (54)$$

is reproduced at the output of g_m. And if the imaginary term inside the bracket of (45) were zero, that is, $q_2 = 0$, this would be the totality of phase modulation at the output of the optimum nonlinearity. Since $q_2 = 0$ corresponds to $p(A,\phi)$ being independent of ϕ, the modification of the amplitude modulation at the nonlinearity input would then be according to

$$- \frac{\partial}{\partial A} \ln \left\{\frac{p_a(A)}{A}\right\} , \quad (55)$$

and the maximum deflection is

$$d_m^2 = \frac{1}{4} A_s^2(t) \int dA \, \frac{\left[p_a'(A) - p_a(A)/A\right]^2}{p_a(A)} \, . \tag{56}$$

However, the presence of the extra imaginary term in (45) means that additional amplitude and phase modulations to those given in (55) and (54), respectively, are superposed on the output. Whether this is significant, in practice, will depend on a quantitative investigation of the relative sizes of q_1^2 and q_2^2 in maximum deflection (46).

SIZE OF NEGLECTED TERM

When we expressed the difference of mean outputs from nonlinearity g in the form (34), we discarded the second term as being of an undesirable form. To see how this neglected term compares with the retained term, in terms of magnitude, we need to compare $|z_a - iz_b|$ with $|z_a + iz_b|$. For the optimum nonlinearity g_m in (45), we have, from (35) and (36),

$$\left|z_a - iz_b\right|_m = \iint dA \, d\phi \, \frac{q_1^2(A,\phi) + q_2^2(A,\phi)}{p(A,\phi)} \, , \tag{57}$$

$$\left|z_a + iz_b\right|_m = \left|\iint dA \, d\phi \, \exp(i2\phi) \, \frac{\left[q_1(A,\phi) + iq_2(A,\phi)\right]^2}{p(A,\phi)}\right| =$$

$$= \left|\iint dA \, d\phi \, \exp(i2\phi) \, \frac{q_1^2(A,\phi) - q_2^2(A,\phi) + i2 \, q_1(A,\phi) \, q_2(A,\phi)}{p(A,\phi)}\right| . \tag{58}$$

Thus, the $\exp(i2\phi)$ term chops up the ϕ integral, since 2ϕ ranges

over 4π, leading to a small value for (58). In addition, the q_1^2 and q_2^2 terms cancel each other in (58), whereas they add in (57). It can also be observed that (57) is identical with the maximum deflection (46), except for factor $A_s^2(t)/4$. Some alternative forms to (57) and (58) in rectangular coordinates are given in appendix D.

Of course, in all cases,

$$\left|z_a + iz_b\right|_m \le \iint dA\, d\phi\, \left|\exp(i2\phi)\right| \frac{\left|q_1(A,\phi)+iq_2(A,\phi)\right|^2}{p(A,\phi)} =$$

$$= \left|z_a - iz_b\right|_m ; \tag{59}$$

however, it is expected that we will have

$$\left|z_a + iz_b\right|_m << \left|z_a - iz_b\right|_m \tag{60}$$

in most practical cases. As an example, the Gaussian noise considered in (48)-(50) yields

$$\left|z_a - iz_b\right|_m = \frac{2}{\sigma_n^2},$$

$$\left|z_a + iz_b\right|_m = \left|\int_0^{+\infty} dA \int_{-\pi}^{\pi} d\phi\, \exp(i2\phi)\, \frac{A^3 \exp\left[-A^2 \big/ \left(2\sigma_n^2\right)\right]}{2\pi\sigma_n^6}\right| = 0. \tag{61}$$

Thus, neglecting the second term in (34) is justified, both in terms of physical interpretation and in terms of magnitude of contribution.

OPTIMUM NONLINEARITY FOR PHASE CHANGES

It is hard to conceive of an independent physical narrowband noise process for which the phase $\phi_n(t)$ would not be uniformly distributed over a 2π interval. In that case, (51) is relevant, with $p_b(\phi)$ constant of value $(2\pi)^{-1}$. Then the second term in optimum nonlinearity (52) is absent, as is the additive term in (53); see also (55) and (56).

NONLINEAR PROCESSING

In this situation, it may be advantageous to resort to additional processing of the phase changes between adjacent time samples of input $x(t)$; regular phase changes would occur for a frequency-shifted narrowband process, such as encountered in FSK communication. In particular, we consider nonlinear processing of adjacent time samples of the received waveform, namely,

$$A_x(t),\ \phi_x(t),\ A_x(t-\Delta),\ \phi_x(t-\Delta), \tag{62}$$

where Δ is the time between samples; see also (13) and (14). Thus, the output of the nonlinearity is generalized from (15) to

$$y(t) = h\{A_x(t),\ \phi_x(t),\ A_x(t-\Delta),\ \phi_x(t-\Delta)\}\ , \tag{63}$$

where h is an arbitrary complex nonlinear transformation of four real variables.

INPUT NOISE STATISTICS

The required statistical information about the input noise process is now the joint probability density function of noise quantities

$$A_n(t),\ \phi_n(t),\ A_n(t-\Delta),\ \phi_n(t-\Delta)\ . \tag{64}$$

We denote these random variables by

$$A_n,\ \phi_n,\ \bar{A}_n,\ \bar{\phi}_n, \tag{65}$$

respectively, and presume that their joint probability density function has the form

$$p_a(\bar{A}_n)\ p_b(\bar{\phi}_n)\ p_a(A_n)\ p_c(\phi_n - \bar{\phi}_n)\ . \tag{66}$$

That is, all the random variables (65) are statistically independent except that $\phi_n = \phi_n(t)$ depends on $\bar{\phi}_n = \phi_n(t-\Delta)$; thus, probability density function p_c can be expected to peak at a point(s) related to the frequency shift(s) of the noise carrier.

MEAN OUTPUT

The mean output from the general nonlinearity, for signal present, is

$$\overline{y_1(t)} = \overline{h\{A_x(t),\ \phi_x(t),\ A_x(t-\Delta),\ \phi_x(t-\Delta)\}} =$$

$$= \iiiint dA_n\ d\phi_n\ d\bar{A}_n\ d\bar{\phi}_n\ p_a(\bar{A}_n)\ p_b(\bar{\phi}_n)\ p_a(A_n)\ p_c(\phi_n-\bar{\phi}_n)\ \times$$

$$\times\ h\{|A_n \exp(i\phi_n) + A_s \exp(i\phi_s)|,\ \arg\{A_n \exp(i\phi_n) + A_s \exp(i\phi_s)\},$$

$$|\bar{A}_n \exp(i\bar{\phi}_n) + \bar{A}_s \exp(i\bar{\phi}_s)|,\ \arg\{\bar{A}_n \exp(i\bar{\phi}_n) + \bar{A}_s \exp(i\bar{\phi}_s)\}\}, \quad (67)$$

where we suppressed the time dependence of the signal terms (for now) by using the notation

$$A_s = A_s(t),\ \phi_s = \phi_s(t),\ \bar{A}_s = A_s(t-\Delta),\ \bar{\phi}_s = \phi_s(t-\Delta)\ . \quad (68)$$

Now make the changes of variables (using appendix A)

$$A \exp(i\phi) = A_n \exp(i\phi_n) + A_s \exp(i\phi_s)\ ,$$

$$\bar{A} \exp(i\bar{\phi}) = \bar{A}_n \exp(i\bar{\phi}_n) + \bar{A}_s \exp(i\bar{\phi}_s)\ , \quad (69)$$

to obtain mean output

$$\overline{y_1(t)} = \iiiint dA\ d\phi\ d\bar{A}\ d\bar{\phi}\ \frac{A\ \bar{A}}{\mu\ \bar{\mu}}\ p_a(\bar{\mu})\ p_b(\bar{\theta})\ \times$$

$$\times\ p_a(\mu)\ p_c(\theta-\bar{\theta})\ h\{A,\phi,\bar{A},\bar{\phi}\}\ , \quad (70)$$

where

$$\mu = \mu(A,\phi) = |A \exp(i\phi) - A_s \exp(i\phi_s)| ,$$

$$\overline{\mu} = \overline{\mu}(\overline{A},\overline{\phi}) = |\overline{A} \exp(i\overline{\phi}) - \overline{A}_s \exp(i\overline{\phi}_s)| ,$$

$$\theta = \theta(A,\phi) = \arg\{A \exp(i\phi) - A_s \exp(i\phi_s)\} ,$$

$$\overline{\theta} = \overline{\theta}(\overline{A},\overline{\phi}) = \arg\{\overline{A} \exp(i\overline{\phi}) - \overline{A}_s \exp(i\overline{\phi}_s)\} . \tag{71}$$

This result, (70), is exact.

SMALL INPUT SIGNAL-TO-NOISE RATIO

For small signal-to-noise ratios, we can now expand the quantities in (71) in power series in A_s and $\overline{A}_s$ through linear terms. Reference to (23) yields

$$\left.\begin{aligned} \mu &\sim A - A_s \cos(\phi_s - \phi) \\ \overline{\mu} &\sim \overline{A} - \overline{A}_s \cos(\overline{\phi}_s - \overline{\phi}) \\ \theta &\sim \phi - \frac{A_s}{A} \sin(\phi_s - \phi) \\ \overline{\theta} &\sim \overline{\phi} - \frac{\overline{A}_s}{\overline{A}} \sin(\overline{\phi}_s - \overline{\phi}) \end{aligned}\right\} \begin{aligned} &\text{for } A_s \ll A \\ &\text{and } \overline{A}_s \ll \overline{A} . \end{aligned} \tag{72}$$

Therefore

$$\frac{A}{\mu} \sim \frac{A}{A - A_s \cos(\phi_s - \phi)} \sim 1 + \frac{A_s}{A} \cos(\phi_s - \phi) ,$$

$$\frac{\bar{A}}{\bar{\mu}} \sim 1 + \frac{\bar{A}_s}{\bar{A}} \cos(\bar{\phi}_s - \bar{\phi}) , \qquad (73)$$

where we used (72) and (27). Substitution in (70) gives approximation

$$\overline{y_1(t)} \sim \iiiint dA\, d\phi\, d\bar{A}\, d\bar{\phi} \left[1 + \frac{A_s}{A} \cos(\phi_s - \phi)\right]\left[1 + \frac{\bar{A}_s}{\bar{A}} \cos(\bar{\phi}_s - \bar{\phi})\right] \times$$

$$\times\, p_a\left(\bar{A} - \bar{A}_s \cos(\bar{\phi}_s - \bar{\phi})\right)\, p_b\left(\bar{\phi} - \frac{\bar{A}_s}{\bar{A}} \sin(\bar{\phi}_s - \bar{\phi})\right)\, p_a\left(A - A_s \cos(\phi_s - \right.$$

$$\left. -\phi)\right)\, p_c\left(\phi - \bar{\phi} - \frac{A_s}{A} \sin(\phi_s - \phi) + \frac{\bar{A}_s}{\bar{A}} \sin(\bar{\phi}_s - \bar{\phi})\right)\, h\{A,\phi,\bar{A},\bar{\phi}\}. \qquad (74)$$

Again, however, as noted below (28), the $1/A$ and $1/\bar{A}$ dependencies are incorrect for small A or $\bar{A}$; we must be prepared to modify or discard the $1/A$ dependency in some cases where infinities in behavior arise. The discussion in appendix C is again very relevant.

Now we could expand mean output (74) through linear terms in A_s and $\bar{A}_s$. However, since we have only one nonlinearity h{ } to choose, we will not be able to simultaneously maximize the coefficients of both A_s and $\bar{A}_s$. Instead, we concentrate solely on $A_s = A_s(t)$ and maximize its coefficient; this is consistent with the observation that the output of nonlinearity (63) at time $t-\Delta$ will already have maximized the coefficient of $\bar{A}_s = A_s(t-\Delta)$

when it was the current output. Then, to linear terms in A_s, (74) becomes

$$\overline{y_1(t)} \sim \iiiint dA\, d\phi\, d\bar{A}\, d\bar{\phi} \Big\{ p_a(\bar{A})\, p_b(\bar{\phi})\, p_a(A)\, p_c(\phi - \bar{\phi}) +$$

$$+ \frac{A_s}{A} \cos(\phi_s - \phi)\, p_a(\bar{A})\, p_b(\bar{\phi})\, p_a(A)\, p_c(\phi - \bar{\phi}) -$$

$$- A_s \cos(\phi_s - \phi)\, p_a(\bar{A})\, p_b(\bar{\phi})\, p_a'(A)\, p_c(\phi - \bar{\phi}) -$$

$$- \frac{A_s}{A} \sin(\phi_s - \phi)\, p_a(\bar{A})\, p_b(\bar{\phi})\, p_a(A)\, p_c'(\phi - \bar{\phi}) \Big\}\, h\{A, \phi, \bar{A}, \bar{\phi}\} \quad . \quad (75)$$

The leading term in (75) is $\overline{y_o(t)}$, the noise-only mean output. The remaining terms contain A_s linearly and a combination of $\exp(i\phi_s)$ and $\exp(-i\phi_s)$ terms. *As explained in* (34) et seq., the desired signal term is that containing just $\exp(i\phi_s)$. It is, from (75),

$$\frac{1}{2} A_s \exp(i\phi_s) \iiiint dA\, d\phi\, d\bar{A}\, d\bar{\phi}\, h\{A, \phi, \bar{A}, \bar{\phi}\} \exp(-i\phi)\, p_a(\bar{A})\, p_b(\bar{\phi}) \times$$

$$\times \left\{ \frac{p_a(A)}{A}\, p_c(\phi - \bar{\phi}) - p_a'(A)\, p_c(\phi - \bar{\phi}) + i\, \frac{p_a(A)}{A}\, p_c'(\phi - \bar{\phi}) \right\} \quad . \quad (76)$$

VARIANCE OF OUTPUT

At the same time, the variance of the nonlinearity output (63) for noise-only is exactly

$$\mathrm{var}(y_o(t)) = \iiiint dA\,d\phi\,d\bar{A}\,d\bar{\phi}\;\left|h\{A,\phi,\bar{A},\bar{\phi}\}\right|^2 p_a(\bar{A})\;p_b(\bar{\phi})\;p_a(A)\;p_c(\phi-\bar{\phi}) \tag{77}$$

where we have set $\overline{y_o(t)} = 0$ as usual. The deflection is equal to the magnitude-squared value of (76) divided by (77). This deflection is maximized by the optimum nonlinearity

$$h_m\{A,\phi,\bar{A},\bar{\phi}\} = \exp(i\phi)\left[\frac{1}{A} - \frac{p_a'(A)}{p_a(A)} - i\,\frac{1}{A}\,\frac{p_c'(\phi-\bar{\phi})}{p_c(\phi-\bar{\phi})}\right] =$$

$$= -\exp(i\phi)\left[\frac{d}{dA}\ln\left(\frac{p_a(A)}{A}\right) + i\,\frac{1}{A}\,\frac{p_c'(\phi-\bar{\phi})}{p_c(\phi-\bar{\phi})}\right], \tag{78}$$

where we have canceled out common terms involving $p_a(\bar{A})$ and $p_b(\bar{\phi})$. We must again take note that (78) is only an approximation and is not accurate for small A.

OPTIMUM NONLINEARITY

Nonlinearity (78) is identical to (52) except for the replacement of $p_b(\phi)$ by $p_c(\phi-\bar{\phi})$; this agrees with the comment in [1; second paragraph under (10)]. Thus, the optimum nonlinearity h_m is independent of amplitude variable $\bar{A}$ and depends only on difference, $\phi-\bar{\phi}$, of phase variables, except for leading factor $\exp(i\phi)$, which reproduces the phase of the input; see the second argument $\phi_x(t)$ in (63). In order to employ (78), the probability density function p_a of noise amplitude, and the probability density function p_c of noise phase changes between samples, must be determined.

If we define auxiliary functions

$$h_a\{A\} = -\frac{d}{dA}\ln\left(\frac{p_a(A)}{A}\right) ,$$

$$h_c\{\theta\} = -\frac{d}{d\theta}\ln\, p_c(\theta) , \qquad (79)$$

then (78) can be expressed as

$$h_m\{A,\phi,\bar{A},\bar{\phi}\} = \exp(i\phi)\left[h_a\{A\} + i\,\frac{1}{A}\,h_c\{\phi-\bar{\phi}\}\right] , \qquad (80)$$

and the optimum nonlinearity output is, by use of (63), explicitly

$$y(t) = \exp[i\phi_x(t)]\left[h_a\{A_x(t)\} + i\,\frac{h_c\{\phi_x(t) - \phi_x(t-\Delta)\}}{A_x(t)}\right] . \qquad (81)$$

All of these results are predicated on the particular model of

noise statistics as given by (66). Another model for the noise joint probability density function would lead to a different optimum nonlinearity.

When the optimum nonlinearity, (80), is substituted into the deflection, which is the magnitude-squared value of (76) divided by variance (77), the maximum deflection is found to be

$$d_m^2 = \frac{1}{4} A_s^2(t) \left[\int dA \; p_a(A) \; h_a^2\{A\} + \right.$$

$$\left. + \int dA \; \frac{p_a(A)}{A^2} \int d\theta \; p_c(\theta) \; h_c^2\{\theta\} \right] . \qquad (82)$$

This quantity depends only on the probability density functions p_a and p_c; see (64)-(66) and (79). The presence of the $p_a(A)/A^2$ term has been discussed earlier and is not valid for small A; see also appendix C.

EXAMPLE

Suppose that probability density functions

$$p_a(A) = \frac{A}{\sigma_a^2} \exp\left(\frac{-A^2}{2\sigma_a^2}\right) \quad \text{for } A > 0 \ ,$$

$$p_c(\theta) = \left(2\pi\sigma_c^2\right)^{-\frac{1}{2}} \exp\left(\frac{-\theta^2}{2\sigma_c^2}\right) \quad \text{for all } \theta \ . \qquad (83)$$

Then (79) yields

$$h_a\{A\} = \frac{A}{\sigma_a^2} \quad \text{for } A > 0 \ , \qquad h_c\{\theta\} = \frac{\theta}{\sigma_c^2} \ , \qquad (84)$$

and optimum nonlinearity (80) becomes

$$h_m\{A,\phi,\bar{A},\bar{\phi}\} = \exp(i\phi)\left[\frac{A}{\sigma_a^2} + i\,\frac{1}{A}\,\frac{\phi - \bar{\phi}}{\sigma_c^2}\right] . \qquad (85)$$

The maximum deflection follows from (82) as

$$d_m^2 = \frac{A_s^2(t)/2}{\sigma_a^2} + \frac{A_s^2(t)}{4\sigma_c^2} \int_0^{+\infty} \frac{dA}{A^2}\ \frac{A}{\sigma_a^2} \exp\left(\frac{-A^2}{2\sigma_a^2}\right) . \qquad (86)$$

However, the last integral on A does not converge at A = 0; this is an example where the inadequacies of the small-signal approximations in (72) and (73) cannot be ignored, and (86) is useless. The 1/A dependency in (85) and the $1/A^2$ term in (86) are incorrect for small A and must be eliminated in that range.

SUMMARY

The transformation of coordinates in (19) et seq. was performed so that the series expansion of $\overline{y_1(t)}$ could be done in terms of derivatives of noise joint probability density function p, rather than derivatives of nonlinearity g. This allows g to be discontinuous, but presumes that probability density function p is differentiable. An alternative approach based upon an analytic transformation g is given in (E-20) et seq.

The deflection criterion has been based upon a difference of complex mean outputs for arbitrary signal waveform, as given by (30). This philosophy has been explained in figure 5; it takes full advantage of the fact that noise-only mean output $\overline{y_o(t)}$ is a known complex quantity and can be subtracted out. Equivalently, restricting the nonlinear transformation to the class with zero-mean noise-only outputs does not detract from the attainable deflection values.

Even for small input signal amplitudes, the difference in mean outputs, (34), contains a spurious term in addition to the desirable term, for a general nonlinearity g. We have chosen to ignore the undesired term and to concentrate on maximization of the desired one. After pursuing this approach, we returned to a quantitative measure of the size of the undesired term and found that it was generally quite small; see (57)-(61). Thus, our approach was confirmed to be a consistent one. An alternative viewpoint is given in appendix E, where it is shown that

deliberate suppression of this spurious term causes a degradation in the maximum deflection attainable.

In equations (45), (52), (78), and (80), there is a 1/A term in the imaginary part of the "optimum" nonlinearity. This would appear to indicate that the imaginary component is very important for small inputs; see (81) for example. However, we have then violated the assumptions under which these results were derived, such as in (23), (26), (27), (72), (73), and (75). For example, (23) presumes that A is much larger than A_s. What this means is that the true optimum nonlinearity does not really have a 1/A dependence for small A; however, we do not know what the exact dependence is for small A, because our presumptions preclude investigation in that region. In practice, this means that, for small inputs, we must somehow limit the size of the imaginary part of the nonlinearity output, but the exact transition value and behavior is unknown. A discussion of this problem is presented in appendix C, along with an example of its application and illustration of the basic principles.

APPENDIX A. JACOBIAN OF TRANSFORMATION

Suppose we want to make the two-dimensional transformation between polar coordinates r,θ and ρ,ϕ, according to

$$r \exp(i\theta) = \rho \exp(i\phi) + a \exp(ib) , \tag{A-1}$$

where a and b are arbitrary real constants. The Jacobian of this transformation is

$$\frac{\partial(r,\theta)}{\partial(\rho,\phi)} = \begin{vmatrix} \frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \phi} \\ \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial \phi} \end{vmatrix} . \tag{A-2}$$

From (A-1), there follows directly

$$\frac{\partial r}{\partial \rho} \exp(i\theta) + r\, i \exp(i\theta) \frac{\partial \theta}{\partial \rho} = \exp(i\phi) ,$$

$$\frac{\partial r}{\partial \phi} \exp(i\theta) + r\, i \exp(i\theta) \frac{\partial \theta}{\partial \phi} = \rho\, i \exp(i\phi) . \tag{A-3}$$

Equating real and imaginary parts of these two equations, we have

$$\frac{\partial r}{\partial \rho} = \cos(\phi-\theta) , \quad r \frac{\partial \theta}{\partial \rho} = \sin(\phi-\theta) ,$$

$$\frac{\partial r}{\partial \phi} = - \rho \sin(\phi-\theta) , \quad r \frac{\partial \theta}{\partial \phi} = \rho \cos(\phi-\theta) . \tag{A-4}$$

Substituting in (A-2), there follows the desired result

$$\frac{\partial(r,\theta)}{\partial(\rho,\phi)} = \frac{\rho}{r} = \frac{\rho}{|\rho\ \exp(i\phi) + a\ \exp(ib)|} =$$

$$= \frac{\rho}{\left(\rho^2 + a^2 + 2a\rho\ \cos(\phi-b)\right)^{\frac{1}{2}}}\ , \qquad \text{(A-5)}$$

where we used (A-1).

If there were a need to solve for the individual terms in (A-2), they can be obtained from (A-4), by using the real and imaginary parts of (A-1) to eliminate $\cos\theta$ and $\sin\theta$, with the end results

$$\frac{\partial r}{\partial\rho} = \frac{1}{r}\left[\rho + a\ \cos(\phi-b)\right]\ ,$$

$$\frac{\partial r}{\partial\phi} = -\ \frac{a\rho}{r}\ \sin(\phi-b)\ ,$$

$$\frac{\partial\theta}{\partial\rho} = \frac{a}{r^2}\ \sin(\phi-b)\ ,$$

$$\frac{\partial\theta}{\partial\phi} = \frac{\rho}{r^2}\left[\rho + a\ \cos(\phi-b)\right]\ , \qquad \text{(A-6)}$$

with

$$r = \left(\rho^2 + a^2 + 2a\rho\ \cos(\phi-b)\right)^{\frac{1}{2}}\ . \qquad \text{(A-7)}$$

APPENDIX B. INDEPENDENCE OF ADDITIVE CONSTANT

We will show here that the deflection (43) is unchanged if $g\{A \exp(i\phi)\}$ is replaced by $g\{A \exp(i\phi)\} + b$, where b is a complex constant. This is a simple exercise for the denominator of (43), as a direct substitution and expansion immediately reveals that b cancels out everywhere.

For the numerator of (43), we have an additive term (inside the magnitude-squared) of value

$$b \iint dA\, d\phi\, \exp(-i\phi) \left[A \frac{\partial}{\partial A}\left(\frac{p(A,\phi)}{A}\right) - i \frac{\partial}{\partial \phi}\left(\frac{p(A,\phi)}{A}\right)\right] . \qquad \text{(B-1)}$$

However, integration by parts yields*

$$\int dA\, A \frac{\partial}{\partial A}\left(\frac{p(A,\phi)}{A}\right) = - \int dA \frac{p(A,\phi)}{A} \qquad \text{(B-2)}$$

as well as**

$$\int d\phi\, \exp(-i\phi) \frac{\partial}{\partial \phi}\left(\frac{p(A,\phi)}{A}\right) = i \int d\phi\, \exp(-i\phi) \frac{p(A,\phi)}{A} . \qquad \text{(B-3)}$$

Substitution of these two results in (B-1) yields

$$b\left[\int d\phi\, \exp(-i\phi)\, (-1) \int dA \frac{p(A,\phi)}{A} - i \int dA\, i \int d\phi\, \exp(-i\phi) \frac{p(A,\phi)}{A}\right] \qquad \text{(B-4)}$$

which is identically zero. Thus, the additive term dependent on b is zero.

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*Result (B-2) is true if $p(0,\phi) = 0$.

**The 2π periodicity of $p(A,\phi)$ in ϕ is utilized in getting (B-3).

APPENDIX C. BEHAVIOR OF OPTIMUM NONLINEARITY

Some problems with the approximations utilized in (23)-(27), in order to simplify the mean output $\overline{y_1(t)}$, were pointed out in the sequel to (28) and were manifested in the example in (53) by means of the 1/A dependencies for small A. To circumvent these limitations, we will adopt the procedure used just after figure 3 for the lowpass case, namely, investigation of the exact deflection and corresponding optimum nonlinearity with knowledge of signal amplitude $A_s(t)$ and phase $\phi_s(t)$. Again, although physically unrealistic, this approach is informative and does furnish an absolute upper bound on performance.

The starting point is the exact result (21) for the nonlinearity mean output,

$$\overline{y_1(t)} = \iint dA\, d\phi\, \frac{A}{|z|}\, g\{A \exp(i\phi)\}\, p(|z|, \arg(z)) \,, \tag{C-1}$$

where (dropping explicit signal t dependence)

$$z = A \exp(i\phi) - A_s \exp(i\phi_s). \tag{C-2}$$

The noise-only mean output is obtained by setting $A_s = 0$:

$$\overline{y_o(t)} = \iint dA\, d\phi\, g\{A \exp(i\phi)\}\, p(A,\phi) \,. \tag{C-3}$$

The variance of $y_o(t)$ is given by (42).

We now define exact deflection

$$\bar{d}^2(A_s,\phi_s) = \frac{\left|\overline{y_1(t)} - \overline{y_o(t)}\right|^2}{\mathrm{var}(y_o(t))} , \qquad \text{(C-4)}$$

where the dependence on signal parameters is made explicit. Since the absolute scale of nonlinearity g and an additive constant to g do not affect the deflection, we can simplify (C-4) to

$$\bar{d}^2(A_s,\phi_s) = \frac{\left|\iint dA\, d\phi\; g\{A\exp(i\phi)\}\left[\frac{A}{|z|}\, p(|z|,\arg(z)) - p(A,\phi)\right]\right|^2}{\iint dA\, d\phi\, \left|g\{A\exp(i\phi)\}\right|^2 p(A,\phi)} . \qquad \text{(C-5)}$$

By Schwartz's inequality, the optimum nonlinearity (with no approximations, but with assumed knowledge of A_s and ϕ_s) is

$$\bar{g}_e(A,\phi;A_s,\phi_s) = \frac{A}{|z|}\,\frac{p(|z|,\,\arg(z))}{p(A,\phi)} - 1 , \qquad \text{(C-6)}$$

where z is given by (C-2). The corresponding maximum deflection follows from (C-5) as

$$\bar{d}_e^2(A_s,\phi_s) = \iint dA\, d\phi\, \left[\frac{A}{|z|}\, p(|z|,\,\arg(z)) - p(A,\phi)\right]^2 \Big/ p(A,\phi) =$$

$$= \iint dA\, d\phi\, \frac{A^2}{|z|^2}\,\frac{p^2(|z|,\,\arg(z))}{p(A,\phi)} - 1 . \qquad \text{(C-7)}$$

From (C-2), since

$$|z| = \left[A^2 + A_s^2 - 2\,A\,A_s\cos(\phi_s - \phi)\right]^{\frac{1}{2}} , \qquad \text{(C-8)}$$

it follows that

$$|z| \sim A_s \quad \text{as} \quad A \to 0+ \quad (\text{if } A_s \neq 0) \tag{C-9}$$

and, therefore, exact optimum nonlinearity $\bar{g}_e$ in (C-6) has <u>no</u> 1/A dependency for small A (unless probability density function $p(A,\phi)$ approaches zero rapidly for small A), but in fact has a linear dependence on A. That is,

$$\bar{g}_e(A,\phi;A_s,\phi_s) \sim \frac{A}{p(A,\phi)} \frac{p(A_s,\phi_s+\pi)}{A_s} - 1 \quad \text{as } A \to 0+ \ . \tag{C-10}$$

Similarly, the integrand of maximum deflection $\bar{d}_e^2$ in (C-7) has no $1/A^2$ dependence, but in fact, an A^2 dependence for small A:

$$\frac{A^2}{p(A,\phi)} \frac{p^2(A_s,\phi_s+\pi)}{A_s^2} \quad \text{as} \quad A \to 0+ \ . \tag{C-11}$$

These are marked differences in behavior from the approximate results of (45)-(47).

It should also be noted that optimum nonlinearity $\bar{g}_e$ in (C-6) is real. (More precisely, one of the possible optimum nonlinearities is real since complex multiplicative constants can be dropped.) The reason this disagrees with the complex solution in (45) is that A_s and ϕ_s are presumed known in (C-6). When ϕ_s is unknown, then even if (C-6) is developed in a power series in A_s, it is not possible to extract a nonlinearity that is independent of ϕ_s; this information is too deeply embedded in optimum form (C-6). Thus, (C-6) should only be regarded as a guide to good

processing, especially for small A, but otherwise it is not overly useful. The corresponding maximum deflection in (C-7) is probably more useful since it furnishes an absolute upper bound on performance for any nonlinearity. If an approximate result, like (46) or (50) or (53), outperforms (C-7), it is in error and must be modified or discarded. The apparent infinity in (53) at A = 0, for example, is conspicuously wrong; reference to (C-11) indicates that the true near-origin behavior is significantly different.

The philosophy in this appendix is very different from that utilized in (34)-(39). There, a desired type of signal term was identified up front, while the nonlinearity characteristic g was still arbitrary; then, that particular type of term was maximized by choice of g. Here, the entire nonlinearity output difference of means was maximized without any type of term being designated as desired. Thus, we should expect to be able to realize a larger deflection in this latter case since no terms have been suppressed or ignored. The only problem with this approach is that, after the maximization, it is not generally possible to extract a meaningful nonlinear device that is independent of the input signal values of amplitude and phase.

An example to illustrate these points is furnished by Gaussian narrowband noise as in (48):

$$p(A,\phi) = \frac{A}{2\pi\sigma_n^2} \exp\left(-\frac{A^2}{2\sigma_n^2}\right) \quad \text{for } A > 0,\ |\phi| < \pi . \qquad \text{(C-12)}$$

Then, (C-6) and (C-8) immediately yield optimum nonlinearity

$$\bar{g}_e(A,\phi;A_s,\phi_s) = \exp\left[\frac{A\ A_s}{\sigma_n^2}\cos(\phi_s - \phi) - \frac{A_s^2}{2\sigma_n^2}\right] - 1\ . \qquad \text{(C-13)}$$

For small A_s, this behaves according to

$$\bar{g}_e(A,\phi;A_s,\phi_s) \sim \frac{A\ A_s}{\sigma_n^2}\cos(\phi_s - \phi) \quad \text{as } A_s \to 0\ . \qquad \text{(C-14)}$$

Even though there is a linear term in A_s, which could be factored out, the remaining nonlinearity, namely $\frac{A}{\sigma_n^2}\cos(\phi_s - \phi)$, depends on ϕ_s. It is now <u>too</u> <u>late</u> to express

$$\frac{A\ A_s}{\sigma_n^2}\cos(\phi_s - \phi) =$$

$$= \frac{1}{2\sigma_n^2}\left[A\exp(-i\phi)\ A_s\ \exp(i\phi_s) + A\exp(i\phi)\ A_s\ \exp(-i\phi_s)\right] \qquad \text{(C-15)}$$

and to drop the $A_s \exp(-i\phi_s)$ term as being undesired since this component has been an integral part of the maximization of deflection (C-5). In fact, if we drop that term in (C-15), we are left with nonlinearity

$$A\exp(-i\phi)\ \frac{A_s}{2\sigma_n^2}\exp(i\phi_s)\ , \qquad \text{(C-16)}$$

which can be modified to $A\exp(-i\phi)$ since complex multiplicative factors on the nonlinearity are irrelevant. However, $A\exp(-i\phi)$

manifestly has the wrong phase behavior; see (49). Thus, this series of (late) approximations and replacements can lead to a nonsense processor and must be avoided.

The maximum deflection for this example is obtained by substituting (C-12) in (C-7):

$$\bar{d}_e^2(A_s,\phi_s) = \int_0^{+\infty} dA \int_{-\pi}^{\pi} d\phi \, \frac{A}{2\pi\sigma_n^2} \exp\left[-\frac{1}{\sigma_n^2}\left(\frac{1}{2}A^2 + A_s^2 - 2AA_s \cos(\phi_s-\phi)\right)\right]-1$$

$$= \frac{1}{\sigma_n^2}\int_0^{+\infty} dA \, A \exp\left[-\frac{A^2}{2\sigma_n^2} - \frac{A_s^2}{\sigma_n^2}\right] I_o\left(\frac{2AA_s}{\sigma_n^2}\right) - 1 = \exp\left(\frac{A_s^2}{\sigma_n^2}\right) - 1 \, , \quad \text{(C-17)}$$

where we used [3; 6.631 4]. Observe that this quantity is independent of signal phase ϕ_s. For small input signal-to-noise ratio, this becomes

$$\bar{d}_e^2(A_s,\phi_s) \sim \frac{A_s^2}{\sigma_n^2} \quad \text{as} \quad \frac{A_s}{\sigma_n} \to 0 \, . \quad \text{(C-18)}$$

This latter approximation is twice as large as (50) and is due to the fact that, here, we have retained <u>all</u> the signal terms at the nonlinearity output, whereas the method leading to (50) discarded one of the two possible terms. See the discussion immediately after (C-11).

For some purposes, it may be more useful to express the above relations in terms of the joint probability density function of the in-phase and quadrature components of the narrowband noise rather than the amplitude and phase. Thus, if we let w(u,v) be

the joint probability density function of

$$u_n + iv_n = A_n \exp(i\phi_n) \ , \tag{C-19}$$

then the joint probability density function p of amplitude and phase is given by

$$p(A,\phi) = A \ w(A \cos\phi, \ A \sin\phi) \ . \tag{C-20}$$

APPENDIX D. ALTERNATIVE FORMS IN RECTANGULAR COORDINATES

In (C-19) and (C-20), the joint probability density function $w(u,v)$ of the in-phase and quadrature components of the input noise,

$$u_n + iv_n = A_n \exp(i\phi_n) \; , \tag{D-1}$$

was introduced; it is related to the joint probability density function p of amplitude and phase by

$$p(A,\phi) = A\, w(A\cos\phi,\, A\sin\phi) \; . \tag{D-2}$$

When we employ this result in (31), there follows

$$\begin{aligned} q_1(A,\phi) &= A[\ \cos\phi\ w_1(A\cos\phi,\, A\sin\phi) + \sin\phi\ w_2(A\cos\phi,\, A\sin\phi)] \; , \\ q_2(A,\phi) &= A[-\sin\phi\ w_1(A\cos\phi,\, A\sin\phi) + \cos\phi\ w_2(A\cos\phi,\, A\sin\phi)] \; , \end{aligned} \tag{D-3}$$

and therefore

$$\begin{aligned} q_1(A,\phi) + iq_2(A,\phi) &= A\exp(-i\phi)\ W(A\cos\phi,\, A\sin\phi) \; , \\ q_1(A,\phi) - iq_2(A,\phi) &= A\exp(i\phi)\ W^*(A\cos\phi,\, A\sin\phi) \; , \end{aligned} \tag{D-4}$$

where

$$W(u,v) \equiv w_1(u,v) + iw_2(u,v) = \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) w(u,v) \; . \tag{D-5}$$

When (D-4) is employed in (35) and (36), there follows

$$z_a - iz_b = \iint dA\, d\phi\, A\, g\{A \exp(i\phi)\}\, W^*(A \cos\phi,\, A \sin\phi) =$$

$$= \iint du\, dv\, g\{u + iv\}\, W^*(u,v) \tag{D-6}$$

and

$$z_a + iz_b = \iint dA\, d\phi\, A\, g\{A \exp(i\phi)\}\, W(A \cos\phi,\, A \sin\phi) =$$

$$= \iint du\, dv\, g\{u + iv\}\, W(u,v)\ . \tag{D-7}$$

When the optimum nonlinearity, (45), is expressed in the notation of (D-2)-(D-5), we have

$$g_m\{A \exp(i\phi)\} = -\frac{W(A \cos\phi,\, A \sin\phi)}{w(A \cos\phi,\, A \sin\phi)}\ , \tag{D-8}$$

or

$$g_m\{u + iv\} = -\frac{W(u,v)}{w(u,v)}\ . \tag{D-9}$$

When this is utilized in (D-6) and (D-7), the following alternatives to (57)-(59) result:

$$\left(z_a - iz_b\right)_m = -\iint du\, dv\, \frac{|W(u,v)|^2}{w(u,v)} = -\iint du\, dv\, \frac{w_1^2(u,v) + w_2^2(u,v)}{w(u,v)}\ ,$$

$$\left(z_a + iz_b\right)_m = -\iint du\, dv\, \frac{W^2(u,v)}{w(u,v)} = -\iint du\, dv\, \frac{\left[w_1(u,v) + iw_2(u,v)\right]^2}{w(u,v)}$$

$$= -\iint du\, dv\, \frac{w_1^2(u,v) - w_2^2(u,v) + i2\, w_1(u,v)\, w_2(u,v)}{w(u,v)}\ . \tag{D-10}$$

As an example of these results, suppose that joint probability density function

$$w(u,v) = f(u^2 + v^2); \quad \text{that is, } p(A,\phi) = A\, f(A^2)\ . \tag{D-11}$$

The noise joint probability density function is independent of angle. Then

$$\begin{aligned} w_1(u,v) &= 2\, u\, f'(u^2 + v^2)\ , \\ w_2(u,v) &= 2\, v\, f'(u^2 + v^2)\ , \end{aligned} \tag{D-12}$$

and (D-10) yields

$$\begin{aligned} \left(z_a - i z_b\right)_m &= -4 \iint du\, dv\, (u^2 + v^2)\, \frac{f'^2(u^2 + v^2)}{f(u^2 + v^2)} = \\ &= -\, 8\pi \int_0^{+\infty} dA\, A^3\, \frac{f'^2(A^2)}{f(A^2)}\ , \end{aligned} \tag{D-13}$$

$$\begin{aligned} \left(z_a + i z_b\right)_m &= -4 \iint du\, dv\, (u + iv)^2\, \frac{f'^2(u^2 + v^2)}{f(u^2 + v^2)} = \\ &= -\, 4 \int_{-\pi}^{\pi} d\phi\, \exp(i2\phi) \int_0^{+\infty} dA\, A^3\, \frac{f'^2(A^2)}{f(A^2)} = 0\ . \end{aligned} \tag{D-14}$$

Example (D-11) is a generalization of Gaussian probability density function (48) and (D-14) generalizes (61).

APPENDIX E. CONSTRAINED MAXIMIZATION OF DEFLECTION

When the difference of mean outputs was expanded through linear terms in signal amplitude $A_s(t)$, the end result for arbitrary nonlinearity g was (34):

$$\overline{y_1(t)} - \overline{y_o(t)} = -\frac{1}{2} A_s(t) \exp[i\phi_s(t)] (z_a - iz_b) -$$

$$-\frac{1}{2} A_s(t) \exp[-i\phi_s(t)] (z_a + iz_b) . \quad \text{(E-1)}$$

At that time, we ignored the second term as being of no interest and maximized the first (desired) term; see (39) and (43). Then, we later returned to investigate the relative sizes of these terms in (57)-(61) and appendix D.

Here, we adopt a different viewpoint: we force the second term in (E-1) to be zero and then we maximize the magnitude of the first term by choice of nonlinearity g. More precisely, we maximize deflection (43), subject to integral constraint

$$\iint du\, dv\, g\{u + iv\}\, W(u,v) = 0 \; ; \quad \text{(E-2)}$$

this last equation comes from (D-7) and (D-5).

The first point to observe is that the absolute scale of g does not affect (43) or (E-2). The second is that the same independence is true for an additive complex constant, b, to g. This was proven in appendix B for (43) and follows for (E-2) since

$$\iint du\, dv\, b\, W(u,v) = b \iint du\, dv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) w(u,v) =$$

$$= b \int dv \int du \frac{\partial}{\partial u} w(u,v) + ib \int du \int dv \frac{\partial}{\partial v} w(u,v) = 0 \;, \quad \text{(E-3)}$$

where we used (D-5) and the facts that

$$\int du \frac{\partial}{\partial u} w(u,v) = \Big[w(u,v) + c_1(v) \Big]_{u=-\infty}^{u=+\infty} = 0 \;,$$

$$\int dv \frac{\partial}{\partial v} w(u,v) = \Big[w(u,v) + c_2(u) \Big]_{v=-\infty}^{v=+\infty} = 0 \;. \quad \text{(E-4)}$$

What this means is that, without loss of generality, we can set the complex nonlinearity mean output for noise-alone, $\overline{y_o(t)}$, equal to zero. This results in deflection (44), which can be expressed in the form

$$d^2 = \frac{1}{4} A_s^2(t) \frac{\left| \iint du\, dv\, g\{u + iv\}\, W^*(u,v) \right|^2}{\iint du\, dv\, \left| g\{u + iv\} \right|^2 w(u,v)} \;, \quad \text{(E-5)}$$

where we used (D-4), (D-1), and (D-2). The problem of interest here is to maximize (E-5), subject to constraint (E-2), by choice of nonlinearity g.

Since (E-2) is really two constraints on the real and imaginary parts, we need two real Lagrange multipliers; letting I denote the integral in (E-2), we must add to (E-5) the two real terms

$$\lambda_r\, I_r + \lambda_i\, I_i = \lambda_r \frac{I + I^*}{2} + \lambda_i \frac{I - I^*}{i2} = \frac{1}{2}\left(I\,\lambda^* + I^*\,\lambda\right) , \quad \text{(E-6)}$$

where λ is a complex Lagrange multiplier. Then, the essential quantity we must maximize is

$$Q = \frac{\left|\iint g\, W^*\right|^2}{\iint |g|^2\, w} - \lambda^* \iint g\, W - \lambda \iint g^*\, W^* , \quad \text{(E-7)}$$

where we have dropped irrelevant scale factors and adopted an extremely abbreviated notation for the time being.

If we replace

$$g\{u + iv\} \quad \text{by} \quad g_m\{u + iv\} + \varepsilon\, \eta\{u + iv\} , \quad \text{(E-8)}$$

where $g_m\{\ \}$ is the optimum nonlinearity, $\eta\{\ \}$ is any perturbation, and ε is an arbitrary complex constant, the new value of Q is

$$Q_m + \delta Q = \frac{\iint (g_m + \varepsilon\eta)\, W^* \iint (g_m + \varepsilon\eta)^*\, W}{\iint (g_m + \varepsilon\eta)\,(g_m + \varepsilon\eta)^*\, w} -$$

$$- \lambda^* \iint (g_m + \varepsilon\eta)\, W - \lambda \iint (g_m + \varepsilon\eta)^*\, W^* . \quad \text{(E-9)}$$

Let constants

$$N = \iint g_m\, W^* , \qquad D = \iint |g_m|^2\, w ; \quad \text{(E-10)}$$

N is complex, while D is real. If we set the partial derivative of $Q_m + \delta Q$ with respect to ε^* (for fixed ε) equal to zero, and then set $\varepsilon = 0$, $\varepsilon^* = 0$, we find that we must satisfy

$$\frac{D\, N \iint \eta^* \, W - |N|^2 \iint g_m \, \eta^* \, w}{D^2} - \lambda \iint \eta^* \, W^* = 0 \qquad \text{(E-11)}$$

for all perturbations $\eta = \eta\{u + iv\}$. That is,

$$\iint \eta^* \left[\frac{N}{D}\, W - \frac{|N|^2}{D^2}\, w\, g_m - \lambda\, W^*\right] = 0 \quad \text{for all } \eta . \qquad \text{(E-12)}$$

The solution for the optimum nonlinearity is therefore (dropping irrelevant scale factors)

$$g_m = - \frac{W - \frac{D\lambda}{N} W^*}{w} . \qquad \text{(E-13)}$$

In order to solve for the unknown constants, we must substitute (E-13) into the constraint (E-2). But, first, we define the two additional quantities

$$M = \iint \frac{|W|^2}{w} , \qquad C = \iint \frac{W^2}{w} ; \qquad \text{(E-14)}$$

M is real, while C is complex. Then the substitution yields

$$0 = \iint g_m \, W = - C + \frac{D\lambda}{N} M . \qquad \text{(E-15)}$$

Equation (E-15) can be solved for

$$\frac{D\lambda}{N} = \frac{C}{M} . \qquad \text{(E-16)}$$

Use of this result in (E-13) gives the optimum constrained nonlinearity

$$g_m\{u + iv\} = -\frac{W(u,v) - \frac{C}{M} W^*(u,v)}{w(u,v)} \,, \qquad \text{(E-17)}$$

where constants M and C are given by (E-14). By comparison, the unconstrained optimum nonlinearity did not have the W^* term; see (D-9).

The corresponding maximum deflection is obtained by the use of (E-17) and (E-14) in (E-5):

$$d_m^2 = \frac{1}{4} A_s^2(t)\ M \left(1 - \frac{|C|^2}{M^2}\right) . \qquad \text{(E-19)}$$

The factor

$$1 - \frac{|C|^2}{M^2} = 1 - \left|\frac{\iint du\ dv\ W^2(u,v)/w(u,v)}{\iint du\ dv\ |W(u,v)|^2/w(u,v)}\right|^2 \qquad \text{(E-19)}$$

is the amount by which the constraint (E-2) degrades the attainable deflection; see (46) and (D-4).

ALTERNATIVE APPROACH

If we return to exact mean output $\overline{y_1(t)}$ in (18) and presume now that nonlinearity g is <u>analytic</u>, then a series expansion in $A_s(t)$ yields directly

$$\overline{y_1(t)} - \overline{y_o(t)} =$$

$$\sim A_s(t) \exp[i\phi_s(t)] \iint dA_n \, d\phi_n \, p(A_n, \phi_n) \, g_a'\{A_n \exp[i\phi_n]\} . \quad \text{(E-20)}$$

The subscript a on g denotes that nonlinearity g is now required to be analytic. We immediately see that the second term in (34), of the form $A_s(t) \exp[-i\phi_s(t)]$, is absent. Now use (D-2) and (D-1) to obtain

$$\overline{y_1(t)} - \overline{y_o(t)} = A_s(t) \exp[i\phi_s(t)] \iint du \, dv \, w(u,v) \, g_a'\{u+iv\} . \quad \text{(E-21)}$$

The double integral in (E-21), denoted by I, can be put in two different forms: first, since

$$\frac{\partial}{\partial u}\Big(w(u,v) \, g_a\{u + iv\}\Big) = w_1(u,v) \, g_a\{u + iv\} + w(u,v) \, g_a'\{u + iv\} , \quad \text{(E-22)}$$

then

$$\int du \, w(u,v) \, g_a'\{u + iv\} =$$

$$= \int du \, \frac{\partial}{\partial u}\Big(w(u,v) \, g_a\{u + iv\}\Big) - \int du \, w_1(u,v) \, g_a\{u + iv\} . \quad \text{(E-23)}$$

The first term is

$$\Big[w(u,v)\ g_a\{u + iv\} + c(v)\Big]_{u=-\infty}^{u=+\infty} = 0 \ , \tag{E-24}$$

since we presume that probability density function w(u,v) goes to zero fast enough at $u = \pm\infty$. Therefore, integrating (E-23) on v,

$$I = \iint du\ dv\ w(u,v)\ g_a'\{u+iv\} = - \iint du\ dv\ w_1(u,v)\ g_a\{u+iv\} . \tag{E-25}$$

A similar approach to (E-22), but involving $\partial/\partial v$ instead, yields the alternative expression

$$I = i \iint du\ dv\ w_2(u,v)\ g_a\{u + iv\} \ . \tag{E-26}$$

Since (E-25) and (E-26) are equal, we obtain

$$\iint du\ dv\ W(u,v)\ g_a\{u + iv\} = 0 \ , \tag{E-27}$$

where we used (D-5):

$$W(u,v) = w_1(u,v) + iw_2(u,v) = \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) w(u,v) \ . \tag{E-28}$$

But (E-27) is identical to the constraint (E-2) that we adopted earlier in this appendix. Thus, the assumption of analyticity of nonlinearity g_a automatically realizes the constraint that eliminates the second (undesired) term in (E-1). This is also obvious directly from expansion (E-20).

Since I is given by both (E-25) and (E-26), it is also given by the linear combination

$$I = -\rho\iint du\, dv\, w_1(u,v)\, g_a\{u+iv\}+(1-\rho)i\iint du\, dv\, w_2(u,v)\, g_a\{u+iv\} =$$

$$= - \iint du\, dv\, g_a\{u+iv\}\, [\rho\, w_1(u,v) - i(1-\rho)\, w_2(u,v)] \,, \quad \text{(E-29)}$$

where ρ is any complex constant. Notice that an additive constant to g_a does not affect I, since

$$\int du\, w_1(u,v) = \int du \frac{\partial}{\partial u} w(u,v) = \Big[w(u,v) + c(v)\Big]_{u=-\infty}^{u=+\infty} = 0 \,, \quad \text{(E-30)}$$

and similarly,

$$\int dv\, w_2(u,v) = 0 \,. \quad \text{(E-31)}$$

Then, by reference to (42) and the upper line of (43), the deflection can be expressed as

$$d^2 = A_s^2(t)\, \frac{\left|\iint du\, dv\, g_a\{u+iv\}\Big[\rho\, w_1(u,v) -i(1-\rho)\, w_2(u,v)\Big]\right|^2}{\iint du\, dv\, |g_a\{u+iv\}|^2\, w(u,v)} \,, \quad \text{(E-32)}$$

where we used (E-21), (E-29), (D-2), and (D-1). Now <u>if</u> g_a were unrestricted, the optimum nonlinearity is now

$$g_{am}\{u+iv\} = -2\, \frac{\rho^*\, w_1(u,v) + i(1-\rho^*)\, w_2(u,v)}{w(u,v)} =$$

$$= - \frac{\rho^*\, (W + W^*) + (1-\rho^*)\, (W - W^*)}{w} =$$

$$= - \frac{W(u,v) - (1 - 2\rho^*)\, W^*(u,v)}{w(u,v)} \,, \quad \text{(E-33)}$$

where we used (E-28). But since (E-27) must be satisfied by this candidate nonlinearity g_{am}, we find that

$$1 - 2\rho^* = \frac{C}{M} , \tag{E-34}$$

where C and M are given by (E-14), and (E-33) becomes identically (E-17).

The only thing wrong with this latter alternative approach in (E-20) et seq. is that there is no guarantee that (E-33) yields an analytic nonlinearity g for <u>any</u> ρ. Thus, (E-33) may not be a valid solution. Furthermore, it is unnecessarily restrictive to limit g to being analytic, and the identical optimum nonlinearity, obtained earlier in (E-17), was not restricted to being analytic. In summary, (E-17) is the optimum constrained nonlinearity which eliminates the $A_s(t) \exp[-i\phi_s(t)]$ term, while (D-9) is the optimum nonlinearity which ignores this latter term.

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UNIV OF COLORADO, Boulder, CO (Prof. L. Scharf)	1
UNIV. OF CT, Storrs, CT. (Library and Prof. C. Knapp)	2
UNIV OF FLA, Gainesville, FL (D. Childers)	1
UNIV OF ILLINOIS, Urbana, IL 61801 (Dr. Douglas L. Jones)	1
UNIV OF MICHIGAN, Ann Arbor, MI (EECS Bldg. North Campus))	1
UNIV. OF MINN, Minneapolis, Mn (Prof. M. Kaveh)	1
UNIV. OF NEWCASTLE, Newcastle, NSW, Canada (Prof. A. Cantoni)	1
UNIV. OF QUEENSLAND, St. Lucia, Queensland 4067, Australia (Dr. Boualem Boashash)	1
UNIV. OF RI, Kingston, RI (Prof. G. F. Boudreaux-Bartels, Library, Prof. S. Kay, and Prof. D. Tufts)	4
UNIV. OF ROCHESTER, Rochester, NY (Prof. E. Titlebaum)	1
UNIV. OF SOUTHERN CA., LA. (Prof. William C. Lindsey, Dr. Andreas Polydoros, PHE 414)	2
UNIV. OF STRATHCLYDE, ROYAL COLLEGE, Glasgow, Scotland (Prof. T. Durrani)	1
UNIV. OF TECHNOLOGY, Loughborough, Leicestershire, England (Prof. J. Griffiths)	1
UNIV. OF WASHINGTON, Seattle (Prof. D. Lytle)	1
URICK, ROBERT, Silver Springs, MD	1
US AIR FORCE, Maxwell AF Base, AL (Library)	1
VAN ASSELT, Henrik, USEA S.P.A., La Spezia, Italy	1
VILLANOVA UNIV, Villanova, PA (Prof. Moeness G. Amin)	1
WEAPONS SYSTEMS RESEARCH LAB, Adelaide, Australia	2
WERBNER, A., Medford, MA	1
WESTINGHOUSE ELEC. CORP, OCEANIC DIV, Annapolis, MD (Dr. H. Newman and Dr. H. L. Price)	2
WESTINGHOUSE ELEC. CORP, Waltham, MA (D. Bennett)	1
WILSON JAMES H., San Clemente, CA	1
WOODS HOLE OCEANOGRAPHIC INSTITUTION (Dr. R. Spindel and Dr. E. Weinstein, Library)	3
YALE UNIV. (Library, Prof. P. Schultheiss and Prof. F. Tuteur)	3